Comonadic Account of Feferman-Vaught-Mostowski Theorems

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Let σ be a set of relational symbols with positive arities, we can define a category of σ-structures $\mathcal{R}(\sigma)$:

- Objects are $\mathcal{A} = (A, \{R^{\mathcal{A}}\}_{R\in\sigma})$ where $R^{\mathcal{A}} \subseteq A^r$ for r-ary relation symbol R.
- Morphisms $f : \mathcal{A} \to \mathcal{B}$ are relation preserving set functions $f : A \to B$

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Rightarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

- Embeddings $f\colon A \rightarrowtail B$ are injective functions which reflect relations:

$$R^{\mathcal{A}}(a_1, \ldots, a_r) \Leftarrow R^{\mathcal{B}}(f(a_1), \ldots, f(a_r))$$

Setting for graph theory, database theory, and descriptive complexity

Category theorists look at structures “as they really are”; i.e. up to isomorphism $\mathcal{A} \cong \mathcal{B}$

Model theorists look at structures with the “blurry lens” of a logic $\mathcal{L}$:

$$\mathcal{A} \equiv^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Leftrightarrow \mathcal{B} \vDash \phi$$

$$\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} \equiv^{\mathcal{L}} \mathcal{B}$$

$$\mathcal{A} \Rrightarrow^{\mathcal{L}} \mathcal{B} := \forall \phi \in \mathcal{L}, \mathcal{A} \vDash \phi \Rightarrow \mathcal{B} \vDash \phi$$

For a logic $\mathcal{J}$, $\equiv_{\mathcal{J}}$ may satisfy Feferman-Vaught-Mostowski (FVM) theorems

If $\mathcal{A}_i \equiv_{\mathcal{J}} \mathcal{B}_i$ for all $i \in I$, then

- Products: $\mathcal{A}_1 \times \mathcal{A}_2 \equiv_{\mathcal{J}} \mathcal{B}_1 \times \mathcal{B}_2$ and $\prod_i \mathcal{A}_i \equiv_{\mathcal{J}} \prod_i \mathcal{B}_i$
- Coproducts: $\mathcal{A}_1 + \mathcal{A}_2 \equiv_{\mathcal{J}} \mathcal{B}_1 + \mathcal{B}_2$ and $\coprod_i \mathcal{A}_i \equiv_{\mathcal{L}} \coprod_i \mathcal{B}_i$

For an operation $H \colon \mathcal{C}_1 \times \mathcal{C}_2 \cdots \times \mathcal{C}_n \to \mathcal{D}$ and logics $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$:

$$\mathcal{A}_i \equiv^{\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv^{\mathcal{J}} H(\mathcal{B}_1, \ldots, B_n)$$

with $\mathcal{A}_i, \mathcal{B}_i \in \mathcal{C}_i$ where $\mathcal{C}_i, \mathcal{D}$ are relevant categories of models.

Key ingredient in Courcelle's theorems and other algorithmic metatheorems

How can we prove such statements categorically?

- In every round i, of the k-round game $\mathbf{EF}_k(\mathcal{A}, \mathcal{B})$:
  - Spoiler chooses an element $a_i \in \mathcal{A}$ or $b_i \in \mathcal{B}$
  - Duplicator responds with $b_i \in \mathcal{B}$ or $a_i \in \mathcal{A}$

  Duplicator wins the round if the relation $\gamma_i = \{(a_j, b_j) \mid j \leq i\}$ is a partial isomorphism

Theorem

Duplicator has a winning strategy in $\mathbf{EF}_k(\mathcal{A}, \mathcal{B})$ *iff* $\mathcal{A} \equiv_{\mathcal{L}_k} \mathcal{B}$

One-sided variant: $\mathcal{A} \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}$. No alternation between structures. Partial homomorphism

Bijection variant: $\mathcal{A} \equiv_{\#\mathcal{L}_k} \mathcal{B}$. Duplicator chooses a bijection before Spoiler's choice and responds using bijection

$\#\mathcal{L}_k$ has quantifiers of the from $\exists_{\leq n} x, \exists_{\geq n} x$

Given a σ-structure $\mathcal{A}$, we can create σ-structure $\mathbb{E}_k\mathcal{A}$ on non-empty sequences of elements in A of length $\leq k$

Let $\varepsilon_{\mathcal{A}} : \mathbb{E}_k\mathcal{A} \to \mathcal{A}$ return the last move of the play $[a_1, \ldots, a_n] \mapsto a_n$.

$$R^{\mathbb{E}_k\mathcal{A}}(s_1, \ldots, s_r) \Leftrightarrow s_i \sqsubseteq s_j \text{ or } s_j \sqsubseteq s_i \text{ for } i, j \in [\mathsf{r}] \\ \text{and } R^{\mathcal{A}}(\varepsilon_{\mathcal{A}}(s_1), \ldots, \varepsilon_{\mathcal{A}}(s_r))$$

Comultiplication $\delta \colon \mathbb{E}_k\mathcal{A} \to \mathbb{E}_k\mathbb{E}_k\mathcal{A}$ where

$$\delta([a_1, \ldots, a_n]) = [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]]$$

Theorem (Abramsky+S 21)

- $\mathcal{A} \rightarrow_{\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}$
- $\mathcal{A} \cong_{\mathbf{Kl}(\mathbb{E}_k)} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\#\mathcal{L}_k} \mathcal{B}$ *(with $\mathcal{A}, \mathcal{B}$ finite)*

$\mathcal{A} \rightarrow_{\mathbb{E}_k} \mathcal{B}$ means there exists a Kleisli morphism $f \colon \mathbb{E}_k\mathcal{A} \rightarrow \mathcal{B}$

$\mathcal{A} \cong_{\mathbf{Kl}(\mathbb{E}_k)} \mathcal{B}$ means there exists a Kleisli isomorphism between $\mathcal{A}$ and $\mathcal{B}$

$\exists^+\mathcal{L}_k$ and $\#\mathcal{L}_k$ as logics *without equality*.

Universe of $\mathcal{A}_1 \uplus \mathcal{A}_2 = \{(i, a_i) \mid i = \{1,2\}, a_i \in A_i\}$ and relations defined in obvious way

$$\begin{aligned} R^{\mathcal{A}_1 \uplus \mathcal{A}_2}((i_1, a_1), \ldots, (i_n, a_n)) \Leftrightarrow & \exists i \in \{1,2\} \forall j \in [n], i_j = i \\ & \text{and } R^{\mathcal{A}_i}(a_1, \ldots, a_n) \end{aligned}$$

For every $\mathcal{A}_1, \mathcal{A}_2$ there are

$$\kappa \colon \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \to \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2$$

$$\kappa([(i_1, a_1), \ldots, (i_n, a_1)] = \begin{cases} [a_j \mid i_j = 1] & \text{if } i_n = 1 \\ [a_j \mid i_j = 2] & \text{if } i_n = 2 \end{cases}$$

If $\mathcal{A}_i \to_{\mathbb{E}_k} \mathcal{B}_i$, then $f_i \colon \mathbb{E}_k\mathcal{A}_i \to \mathcal{B}_i$ and $g_i \colon \mathbb{E}_k\mathcal{B}_i \to \mathcal{A}_i$ for $i \in \{1, 2\}$

$$\mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \xrightarrow{\kappa_{\mathcal{A}_1,\mathcal{A}_2}} \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2 \xrightarrow{f_1 \uplus f_2} \mathcal{B}_1 \uplus \mathcal{B}_2$$

So $\mathcal{A}_1 \uplus \mathcal{A}_2 \to_{\mathbb{E}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$ and $\mathcal{A}_1 \uplus \mathcal{A}_2 \Rrightarrow_{\exists^+\mathcal{L}_k} \mathcal{B}_1 \uplus \mathcal{B}_2$

For $\equiv_{\#\mathcal{L}_k}$: if f_i, g_i are inverses for $i \in \{1, 2\}$, then $f_1 \uplus f_2 \circ \kappa, g_1 \uplus g_2 \circ \kappa$ are inverses.

Equivalent to $\kappa \colon \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) \to \mathbb{E}_k\mathcal{A}_1 \uplus \mathbb{E}_k\mathcal{A}_2$ being Kleisli law

$$\varepsilon_{\mathcal{A}_1} \uplus \varepsilon_{\mathcal{A}_2} = \kappa \circ \varepsilon_{\mathcal{A}_1 \uplus \mathcal{A}_2} \quad \delta_{\mathcal{A}_1} \uplus \delta_{\mathcal{A}_2} \circ \kappa = \kappa \circ \mathbb{E}_k\kappa \circ \delta_{\mathcal{A}_1 \uplus \mathcal{A}_2}$$

Theorem

Given

- *operation* $H\colon C_1 \times \cdots \times C_n \to D$,
- *comonads* $\mathbb{C}_1, \ldots, \mathbb{C}_n, \mathbb{D}$ *capturing logics* $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$
- *Kleisli law* $\kappa\colon \mathbb{D}(H(A_1, \ldots, A_n)) \to H(\mathbb{C}_1(A_1), \ldots, \mathbb{C}_n(A_n))$

Then:

$$\mathcal{A}_i \Rrightarrow_{\exists^+\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \Rrightarrow_{\exists^+\mathcal{J}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

$$\mathcal{A}_i \equiv_{\#\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\#\mathcal{J}} H(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Define semantics for $\mathcal{L}_k$ in terms of $\mathbf{EM}(\mathbb{E}_k)$

$\mathbf{EM}(\mathbb{E}_k)$ represent forest-shaped covers of objects in $\mathcal{R}(\sigma)$ of height $\leq k$

Cofree coalgebra functor $F^{\mathbb{E}_k} \colon \mathcal{R}(\sigma) \to \mathbf{EM}(\mathbb{E}_k)$ where $\mathcal{A} \mapsto (\mathbb{E}_k\mathcal{A}, \delta_{\mathcal{A}})$

For $(\mathcal{A}, \alpha \colon \mathcal{A} \to \mathbb{E}_k\mathcal{A})$, we obtain an order $\sqsubseteq_\alpha$ on $\mathcal{A}$ compatible with the relations

$$a \sqsubseteq_\alpha a' \Leftrightarrow \alpha(a) \text{ is prefix of } \alpha(a')$$

Subcategory of paths $(P, \pi) \in \mathcal{P} \subseteq \mathbf{EM}(\mathbb{E}_k)$ where $\sqsubseteq_\pi$ is a finite chain

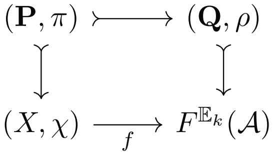
Embeddings $(P, \pi) \rightarrowtail (\mathcal{A}, \alpha)$ pick out paths, and $(P, \pi) \rightarrowtail F^{\mathbb{E}_k}\mathcal{A}$ pick out plays.

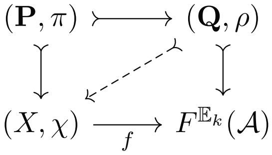
$\mathcal{A} \leftrightarrow_{\mathbb{E}_k} \mathcal{B}$ if there exists a span in $\mathbf{EM}(\mathbb{E}_k)$

$$F^{\mathbb{E}_k}(\mathcal{A}) \xleftarrow{f} (X, \chi) \xrightarrow{g} F^{\mathbb{E}_k}(\mathcal{A})$$

where f, g are

- Pathwise embeddings $e\colon (P, \pi) \rightarrowtail (X, \chi)$ implies $f \circ e\colon (P, \pi) \rightarrowtail F^{\mathbb{E}_k}(\mathcal{A})$
- Open maps, a path which can be extended in the codomain, can be extended in the domain.





Theorem (Abramsky+S 21)

$$\mathcal{A} \leftrightarrow_{\mathbb{E}_k} \mathcal{B} \Leftrightarrow \mathcal{A} \equiv_{\mathcal{L}_k} \mathcal{B}$$

where $\mathcal{L}_k$ is first-order logic up to quantifier rank $\leq k$ without equality.

Compute a span of right type to obtain a FVM theorem for $\uplus$ and $\equiv_{\mathcal{L}_k}$ need a lifting $\hat{\uplus}\colon \mathbf{EM}(\mathbb{E}_k) \times \mathbf{EM}(\mathbb{E}_k) \to \mathbf{EM}(\mathbb{E}_k)$:

$$
\begin{array}{ccc}
 & (X, \chi) & \\
 \swarrow^{f_1 \hat{\uplus} f_2} & & \searrow^{g_1 \hat{\uplus} g_2} \\
F^{\mathbb{E}_k}(\mathcal{A}_1) \hat{\uplus} F^{\mathbb{E}_k}(\mathcal{A}_2) & & F^{\mathbb{E}_k}(\mathcal{B}_1) \hat{\uplus} F^{\mathbb{E}_k}(\mathcal{B}_2) \\
\| & & \| \\
F^{\mathbb{E}_k}(\mathcal{A}_1 \uplus \mathcal{A}_2) & & F^{\mathbb{E}_k}(\mathcal{B}_1 \uplus \mathcal{B}_2)
\end{array}
$$

if f_i, g_i, then $f_1 \uplus f_2, g_1 \uplus g_2$ are OPEs follows from:

(S1) If f_1, f_2 are embeddings, then $f_1 \hat{\uplus} f_2$ is embedding

(S2) $e\colon (P, \pi) \rightarrowtail (\mathcal{A}_1, \alpha_1) \hat{\uplus} (\mathcal{A}_2, \alpha_2)$, then there exists a 'minimal decomposition'

$$e = e_1 \hat{\uplus} e_2 \circ e_0$$

where $e_i\colon (P_i, \pi_i) \rightarrowtail (\mathcal{A}_i, \alpha_i)$ for $i \in \{1, 2\}$

Compute $(\mathcal{A}_1, \alpha_1) \hat{\uplus} (\mathcal{A}_2, \alpha_2)$ as the equaliser in $\mathbf{EM}(\mathbb{E}_k)$

$$(\mathcal{A}_1, \alpha_1) \hat{\uplus} (\mathcal{A}_2, \alpha_2) \xrightarrow{\iota} F^{\mathbb{E}_k}(\mathcal{A}_1 \uplus \mathcal{A}_2) \underset{F(\alpha_1 \uplus \alpha_2)}{\overset{F(\kappa)\circ\delta}{\rightrightarrows}} F^{\mathbb{E}_k}(\mathbb{E}_k \mathcal{A}_1 \uplus \mathbb{E}_k \mathcal{A}_2)$$

- Take the cofree substructure on $F^{\mathbb{E}_k}(\mathcal{A}_1 \uplus \mathcal{A}_2)$
- Substructure compatible with $(\mathcal{A}_i, \alpha_i)$, i.e. the words $[(i_1, a_1), \ldots, (i_n, a_n)] \in F^{\mathbb{E}_k}(\mathcal{A}_1 + \mathcal{A}_2)$:

$$[a_j \mid i_j = 1] \in \mathbf{im}(\alpha_1) \quad [a_j \mid i_j = 2] \in \mathbf{im}(\alpha_2)$$

$\hat{+}$ is a ‘interleaving’ sum of paths in $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_2, \alpha_2)$

Diagram is dual to the quotient construction of a tensor product of vector spaces $V \otimes W$

Dualising ideas about bilinear maps from (commutative) monad theory, we reformulate axioms about $\hat{\uplus}$ to axioms about $\uplus$

1. If $\uplus$ preserves embeddings, then $\hat{\uplus}$ preserves embeddings.
2. For every $(P, \pi) \in \mathcal{P}$, $(\mathcal{A}_i, \alpha_i) \in \mathbf{EM}(\mathbb{E}_k)$ and $f\colon P \to \mathcal{A}_1 + \mathcal{A}_2$ such the following diagram commutes:

$$
\begin{array}{ccccc}
P & & \xrightarrow{\quad\quad\quad f \quad\quad\quad} & & \mathcal{A}_1 + \mathcal{A}_2 \\
\big\downarrow{\scriptstyle \pi} & & & & \big\downarrow{\scriptstyle \alpha_1 \uplus \alpha_2} \\
\mathbb{E}_k(P) & \xrightarrow{\mathbb{E}_k(f)} & \mathbb{E}_k(\mathcal{A}_1 \uplus \mathcal{A}_2) & \xrightarrow{\kappa} & \mathbb{E}_k(\mathcal{A}_1) \uplus \mathbb{E}_k(\mathcal{A}_2)
\end{array}
\qquad (1)
$$

then f has minimal decomposition as $f = e_1 \uplus e_2 \circ e_0$ where $e_i\colon (P_i, \pi_i) \rightarrowtail (\mathcal{A}_i, \alpha_i)$

Theorem

Given n-ary operation H that preserves embeddings, comonads $\mathbb{C}_1, \ldots, \mathbb{C}_n, \mathbb{D}$ capturing $\mathcal{J}_1, \ldots, \mathcal{J}_n, \mathcal{J}$ and $\kappa \colon \mathbb{D}(H(\mathcal{A}_1, \ldots, \mathcal{A}_n)) \to H(\mathbb{C}_1(\mathcal{A}_1), \ldots, \mathbb{C}_n(\mathcal{A}_n))$ satisfying a similar diagram:

$$\mathcal{A}_i \equiv_{\mathcal{J}_i} \mathcal{B}_i \text{ implies } H(\mathcal{A}_1, \ldots, \mathcal{A}_n) \equiv_{\mathcal{J}} H(\mathcal{B}_1, \ldots, B_n)$$

To add equality, we consider a functor $\mathfrak{t}^I \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$ where σ^I has additional binary relation I and $\mathfrak{t}^I(\mathcal{A})$ interprets $I^{\mathfrak{t}^I(\mathcal{A})}$ as equality on $\mathcal{A} \in \mathcal{R}(\sigma)$

Consider $\mathbb{E}_k \circ \mathfrak{t}^I \colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma^I)$ as a relative comonad over $\mathfrak{t}^I$.

As $\mathfrak{t}^I(\mathcal{A}_1 \uplus \mathcal{A}_2) \cong \mathfrak{t}^I(\mathcal{A}_1) \uplus \mathfrak{t}^I(\mathcal{A}_2)$

Study other enrichments such as first-order logic with a connectivity relation **conn** by considering a $\mathfrak{t}^{\mathbf{conn}}$

Products are easiers since right adjoints, such as the cofree-coalgebra functor, preserve limits!

Many other comonads to explore:

- k-variable logic (Abramsky+Dawar+Wang 17)
- modal logic graded by depth
- guarded logics (Abramsky+Marsden 20)
- hybrid/bounded logics (Abramsky+Marsden 21)
- logics with generalised quantifiers (O'Conghaile+Dawar 20)
- logics with restricted conjunction (Montacute+S 22)

All of these are examples of arboreal covers which are studied axiomatically in Abramsky+Reggio 21