Categorical Semantics for Feynman Diagrams

Razin A. Shaikh and Stefano Gogioso

UNIVERSITY OF OXFORD

QUANTINUUM

HASHBERG

Motivation

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- Quantum field theory (QFT) provides the best current model of the universe

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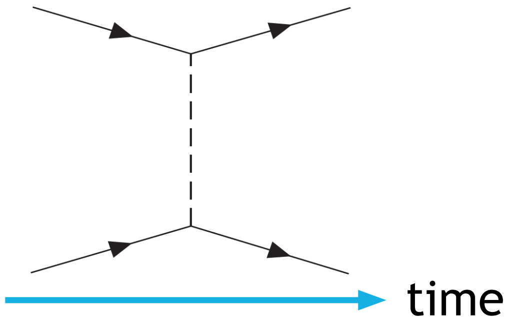

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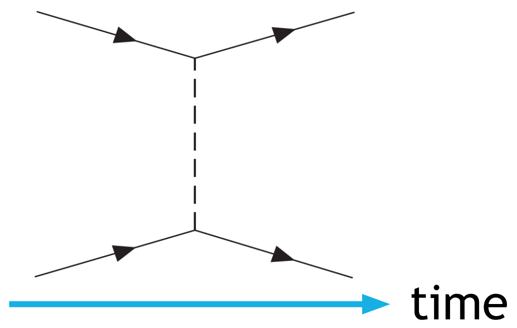

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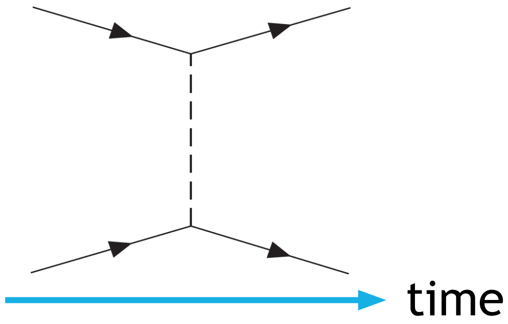

- Quantum field theory (QFT) provides the best current model of the universe
- In QFT, Feynman diagrams represent probability amplitudes of interactions
- How do Feynman diagrams formally arise from field operators?
- Can we compose Feynman diagrams?

Overview

- Categorical quantum fields
- Categorical Feynman diagrams
- Composing Feynman diagrams

Scalar Classical Fields

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Simple harmonic oscillators at each point of momentum space

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To quantise the field, quantize the harmonic oscillators

Quantization

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- We use $^\star\mathrm{fHilb}$ – category of hyperfinite-dimensional Hilbert space
- $^\star\mathrm{fHilb}$ is dagger compact – has cups, caps and spiders
- The results from $^\star\mathrm{fHilb}$ can be transferred to Hilb using a functor

Quantum Harmonic Oscillator

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Annihilation operator

$$a_{\kappa} : \mathcal{H}^{\kappa} \to \mathcal{H}^{\kappa-1}$$
$$|n\rangle \mapsto \sqrt{n}\,|n-1\rangle$$

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Scalar quantum field → *quantum* harmonic oscillators at each point of momentum space Ω

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$$\mathcal{H}^{(\tau)} := \bigotimes_{\underline{p}\in\Omega} \mathcal{H}^{\tau_{\underline{p}}}$$

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Quantum field

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The Fock basis → Particle number for each momentum point:

$$|\beta_{\underline{n}}\rangle := \bigotimes_{\underline{p}\in\Omega} |n_{\underline{p}}\rangle$$

Ingredients required for Feynman diagrams

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- Creation and annihilation operators of the field

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- Feynman propagator

Field Operators

Creation and annihilation operators for fields

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$$a^\dagger_\tau(\underline{p}) : \quad \mathcal{H}^{(\tau)} \longrightarrow \mathcal{H}^{(\tau+\delta_{\underline{p}})}$$

$$|\beta_{\underline{n}}\rangle \quad \mapsto \quad \sqrt{\omega_{ir}^3}\sqrt{n_{\underline{p}}+1}\ |\beta_{\underline{n}+\delta_{\underline{p}}}\rangle$$

$$\delta_{\underline{p}} : \Omega \to \{0,1\}$$

$$\underline{q} \mapsto \delta_{\underline{p},\underline{q}}$$

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$$\begin{aligned} a_\tau(\underline{p}) : \quad \mathcal{H}^{(\tau)} &\longrightarrow \mathcal{H}^{(\tau-\delta_{\underline{p}})} \\ |\beta_{\underline{n}}\rangle &\mapsto \sqrt{\omega_{ir}^3}\sqrt{n_{\underline{p}}}\;|\beta_{\underline{n}-\delta_{\underline{p}}}\rangle \end{aligned}$$

Coherently-controlled Field Operators

We package the field operators into coherently-controlled versions:

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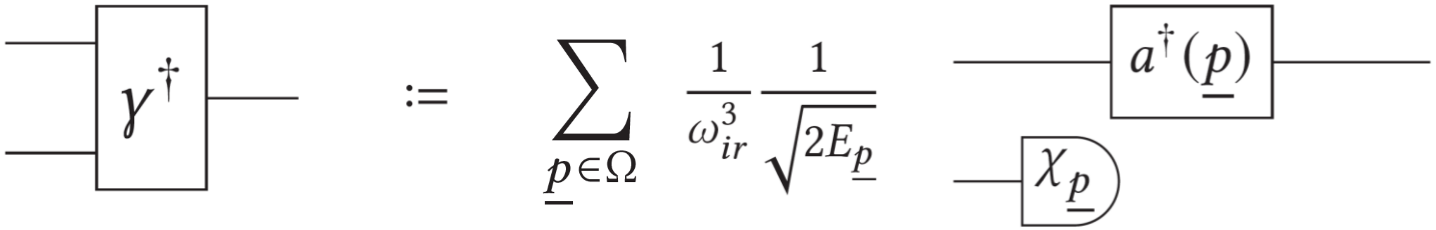

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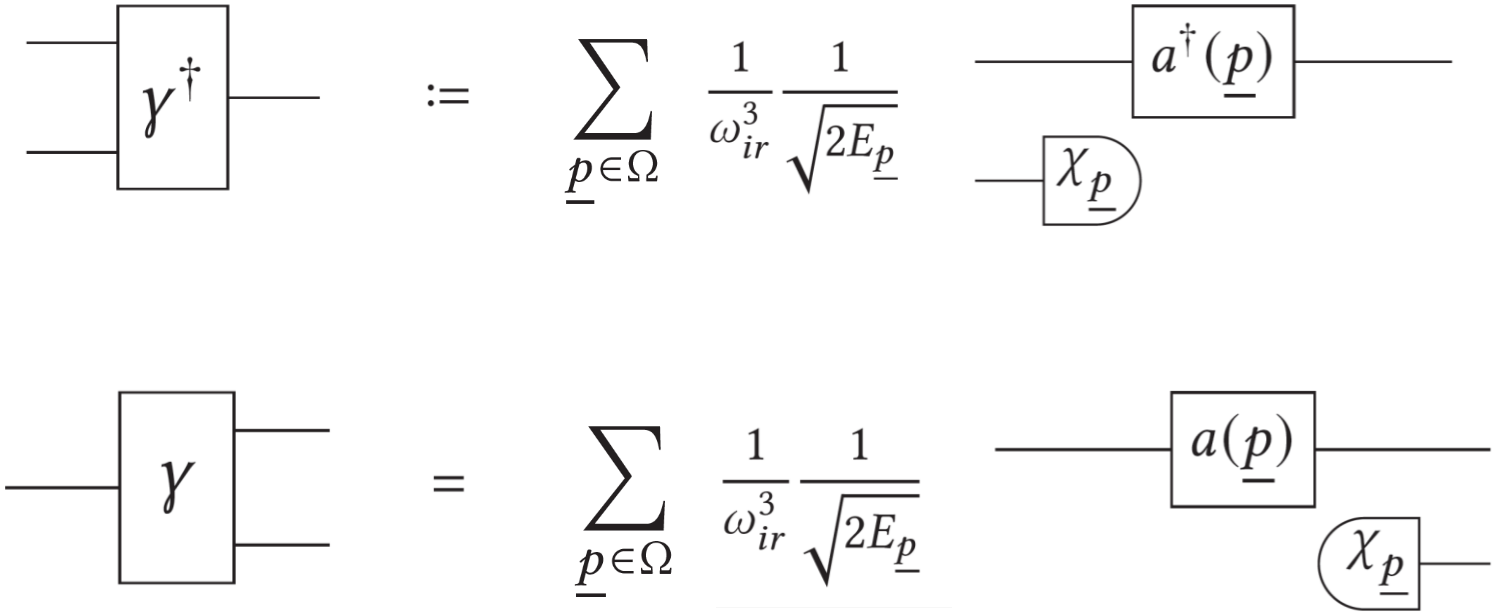
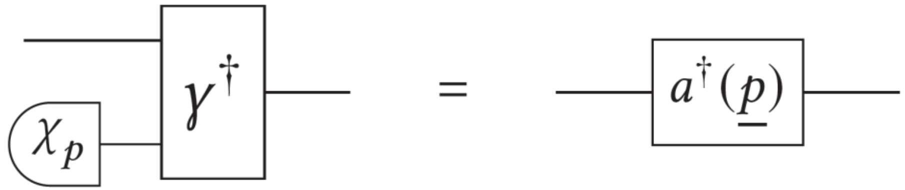
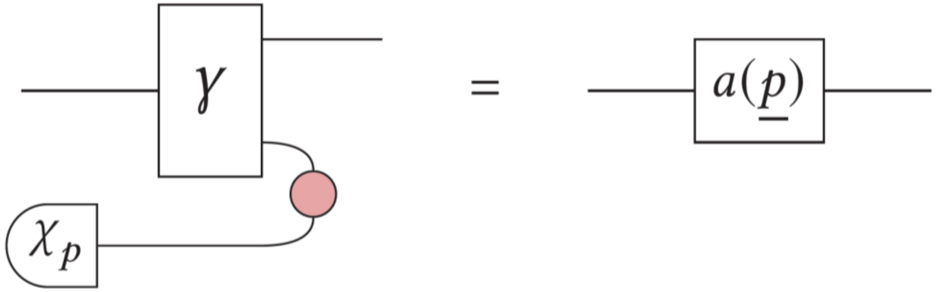


Coherently-controlled Field Operators

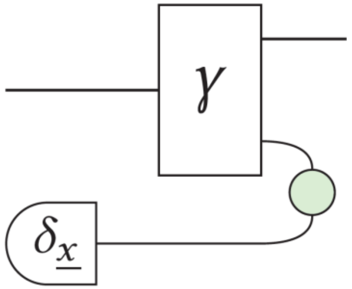
Plug in momentum basis state $\rightarrow$ recover original momentum-space field operators



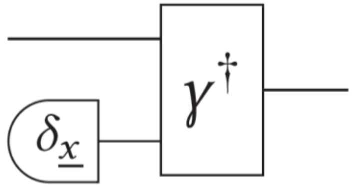


Coherently-controlled Field Operators

Plug in position basis state → Fourier transform of momentum-space operators, i.e. position-space operators



$$= \sum_{\underline{p}} \frac{1}{\omega_{ir}^3} \frac{1}{\sqrt{2E_{\underline{p}}}} a(\underline{p}) e^{i2\pi \underline{p}\cdot\underline{x}} \quad =: \quad \phi^+(\underline{x})$$



$$= \sum_{\underline{p}} \frac{1}{\omega_{ir}^3} \frac{1}{\sqrt{2E_{\underline{p}}}} a^\dagger(\underline{p}) e^{-i2\pi \underline{p}\cdot\underline{x}} \quad =: \quad \phi^-(\underline{x})$$

Feynman propagator

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Probability of a particle travelling from spacetime point x to y

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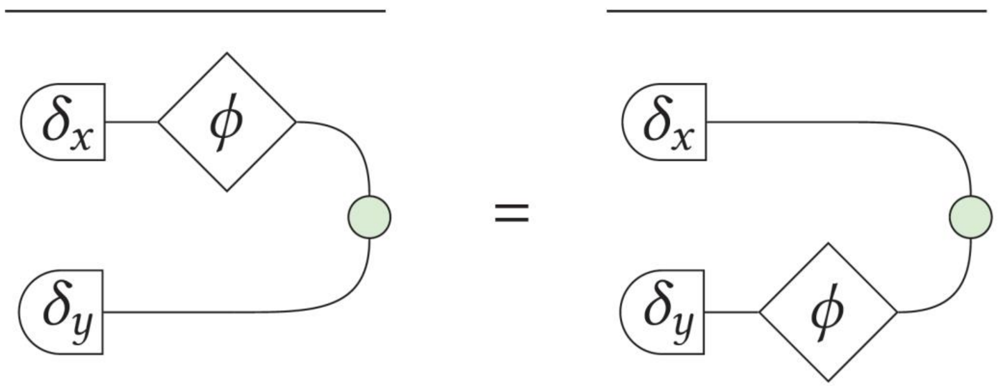
$$\Delta_F(x-y) =$$



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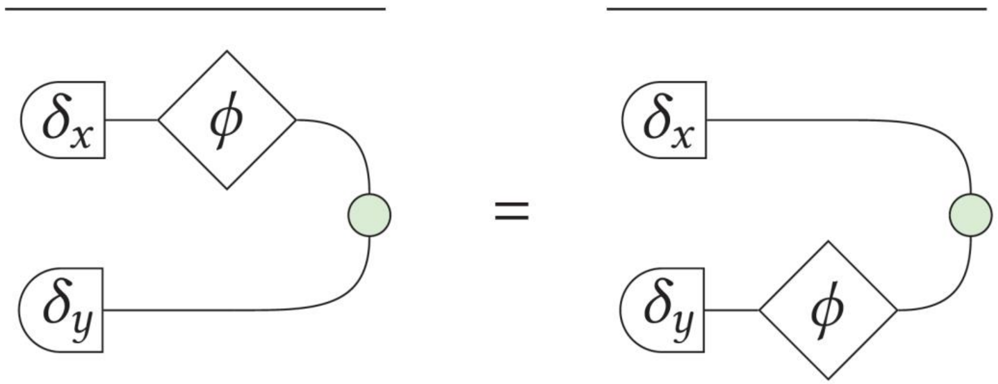

$$\Delta_F(x-y) = \begin{cases} D(x-y) & x_0 > y_0 \\ D(y-x) & y_0 > x_0 \end{cases}$$

$$D(x-y) := \left[\phi^+(x), \phi^-(y)\right]$$

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Scalar Yukawa Theory

- Simplified version of the theory of strong force between nucleons

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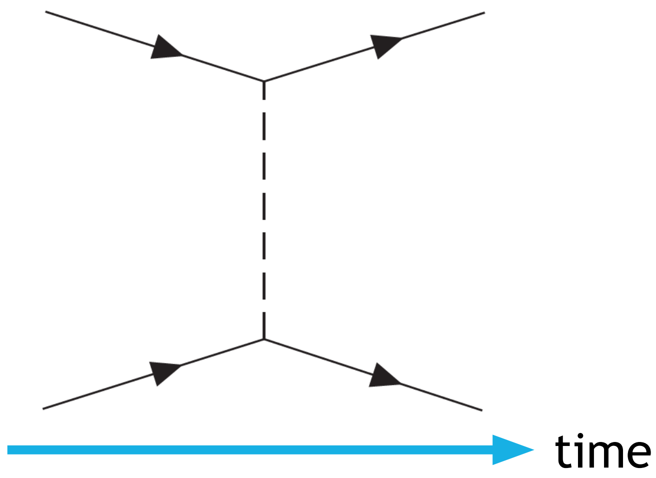
- Simplified version of the theory of strong force between nucleons
- We adopt the following notation:

n_+	nucleons	
n_-	anti-nucleons	
m	mesons	

Example 1: Nucleon-Nucleon Scattering

Interaction of two nucleons mediated by a virtual meson:



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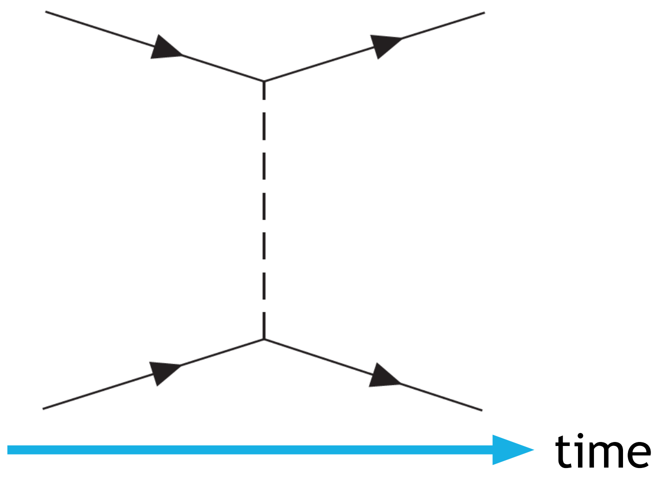

Corresponding Wick's expansion term:

$$\frac{(-ig)^2}{2}\sum_{x_1,x_2}\frac{1}{\omega_{uv}^8}n_+^\dagger(x_1)n_+^\dagger(x_2)n_+(x_1)n_+(x_2)\overbrace{m(x_1)m(x_2)}$$

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⇩

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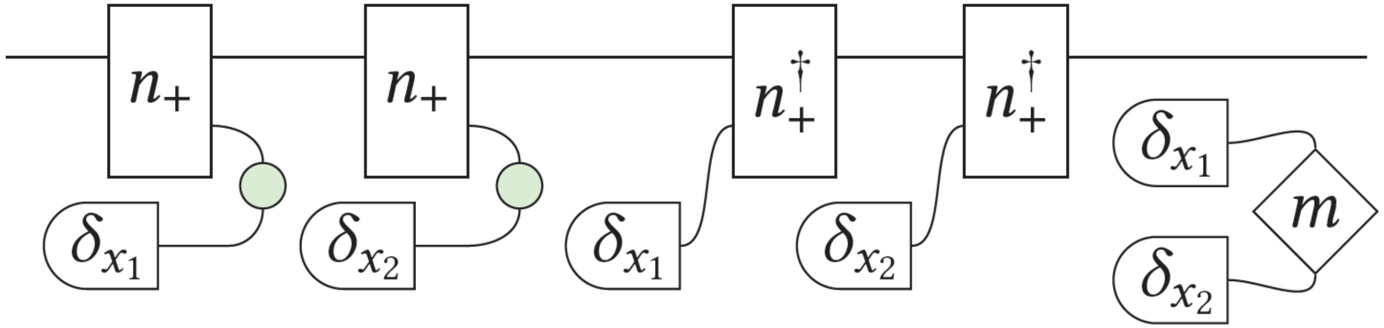

Split and Merge maps

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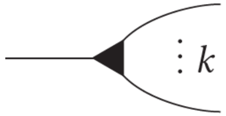

$$: \mathcal{H}^{(\tau)} \to \underbrace{\mathcal{H}^{(\tau)} \otimes \ldots \otimes \mathcal{H}^{(\tau)}}_{k}$$

$$|\underline{n}\rangle \mapsto \sqrt{\langle \underline{n}|\underline{n}\rangle} \sum_{\underline{i} \in \Theta^k_{\underline{n}}} \bigotimes_{j=1}^{k} \frac{1}{\sqrt{\langle \underline{i_j}|\underline{i_j}\rangle}} |\underline{i_j}\rangle$$

$\Theta^k_{\underline{n}}$:= all ways of partitioning particles of $|\underline{n}\rangle$ in k partitions

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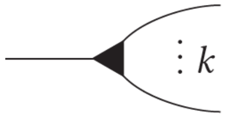

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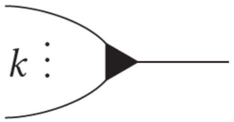

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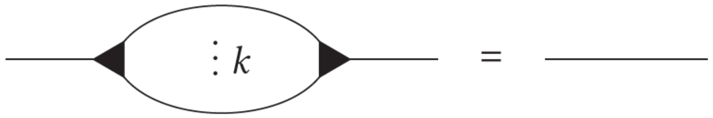

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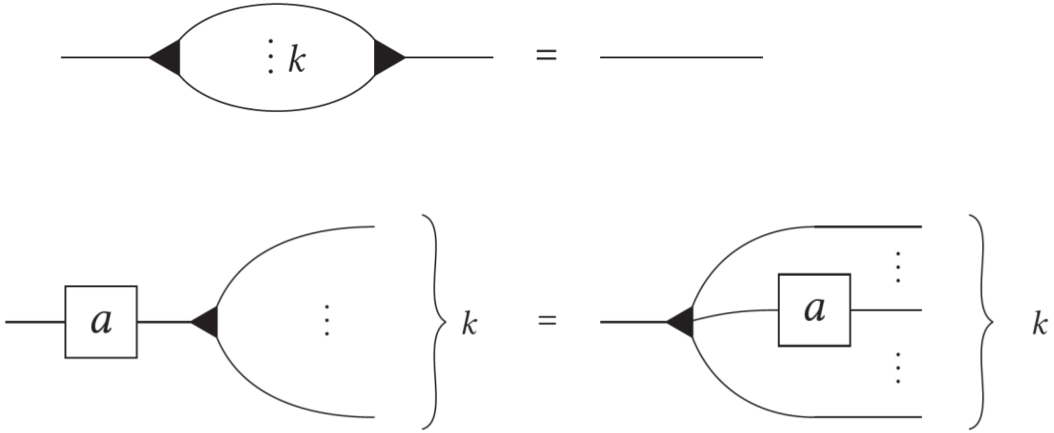

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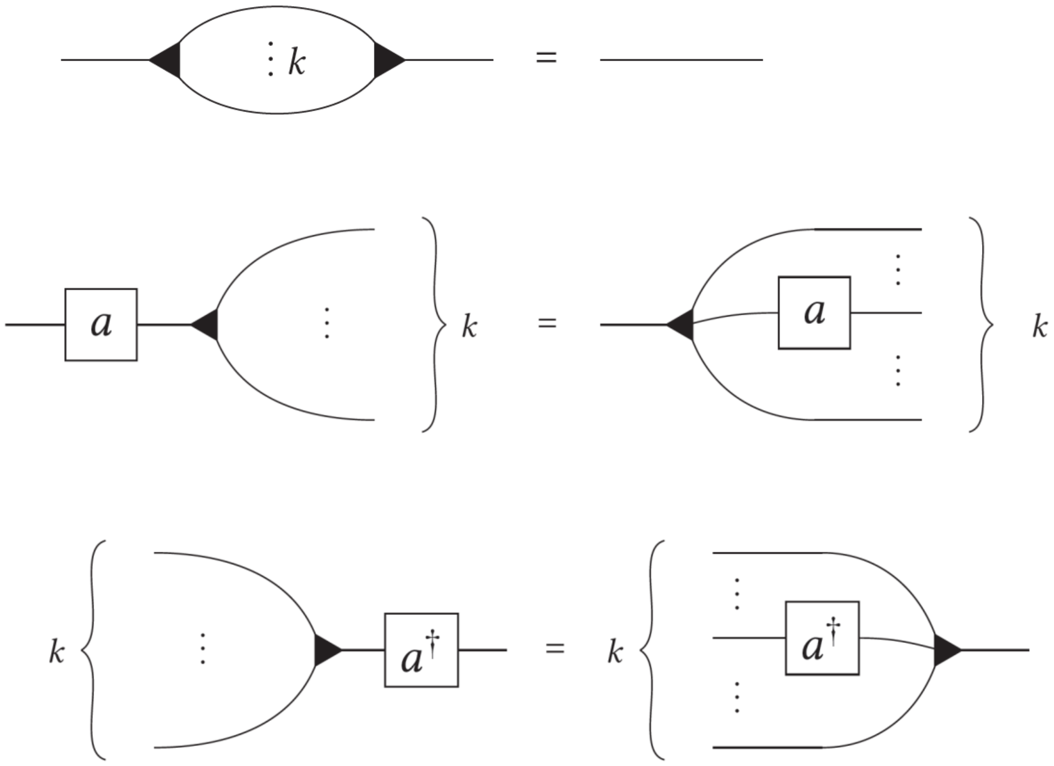

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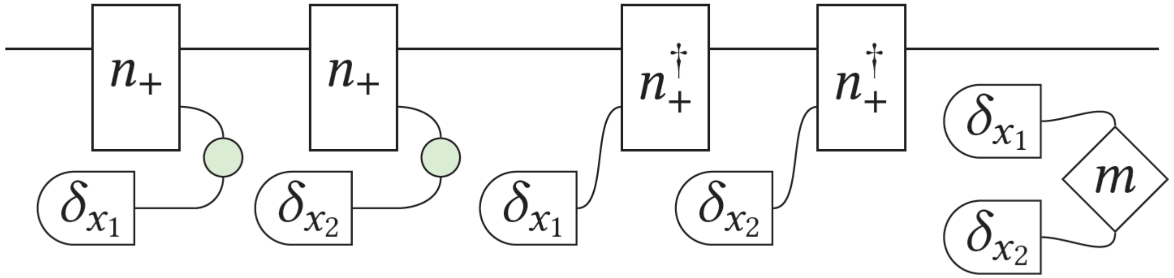
$$\frac{(-ig)^2}{2} \sum_{x_1,x_2} \frac{1}{\omega_{uv}^8}$$



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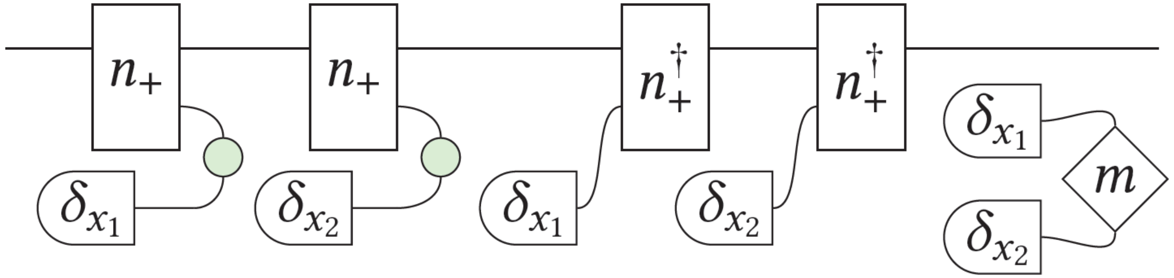

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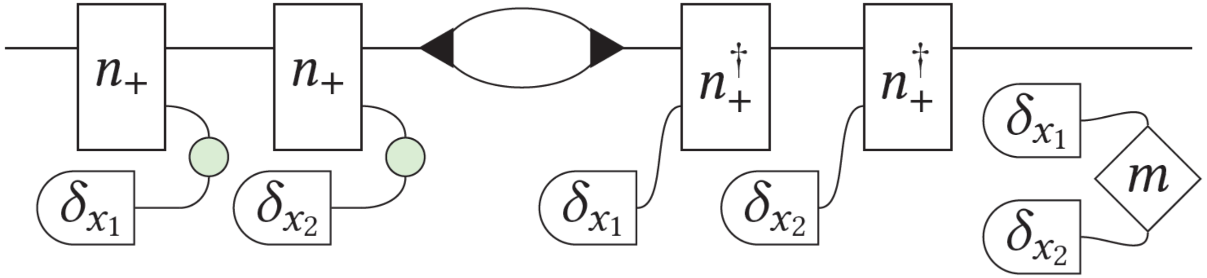

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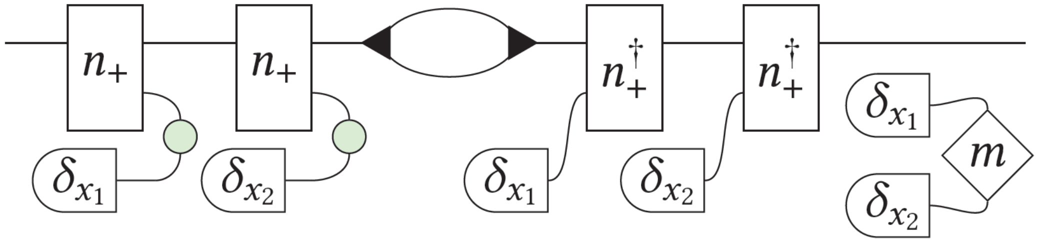

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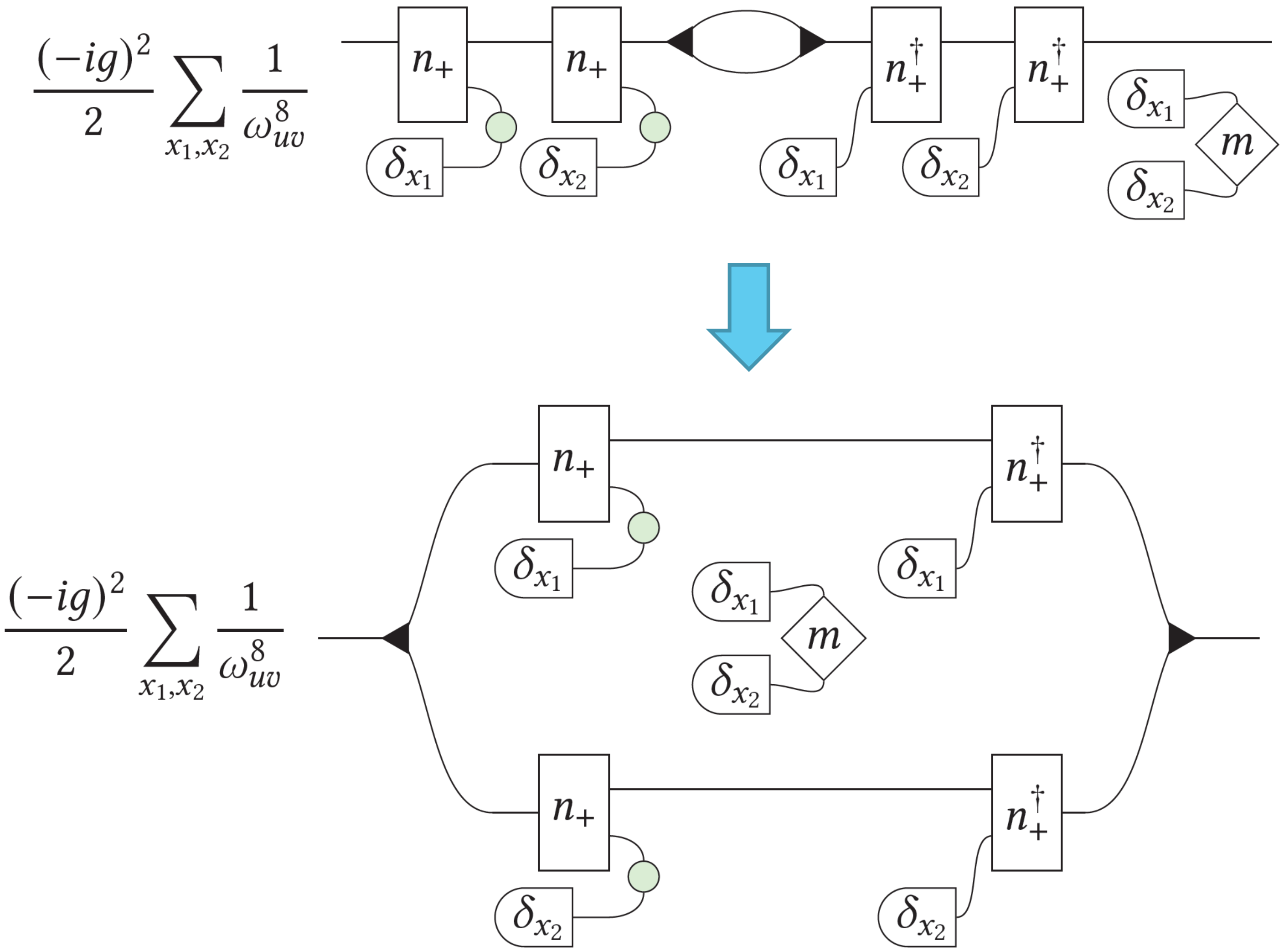

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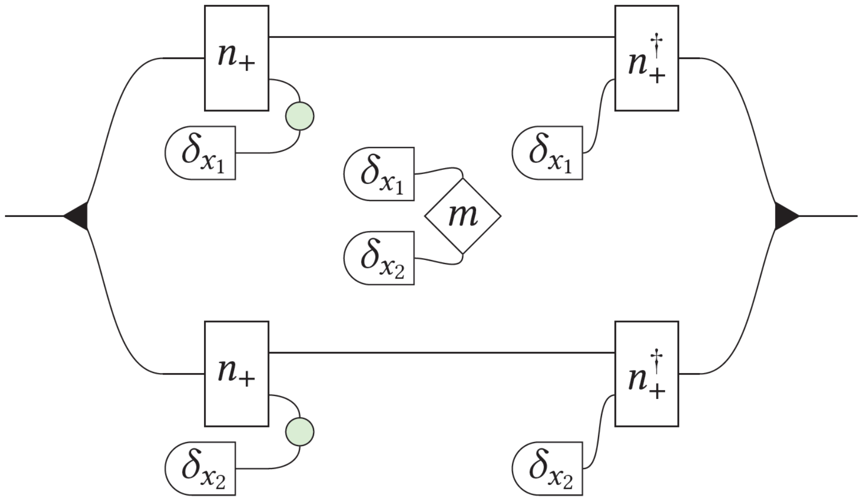

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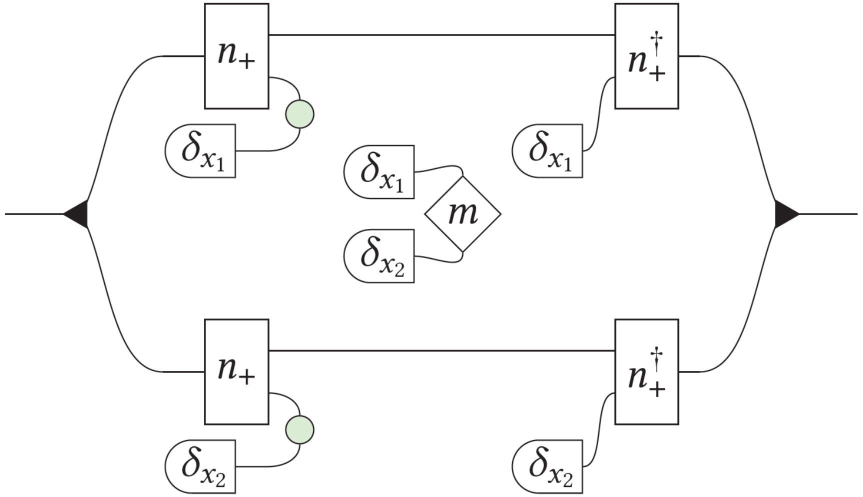

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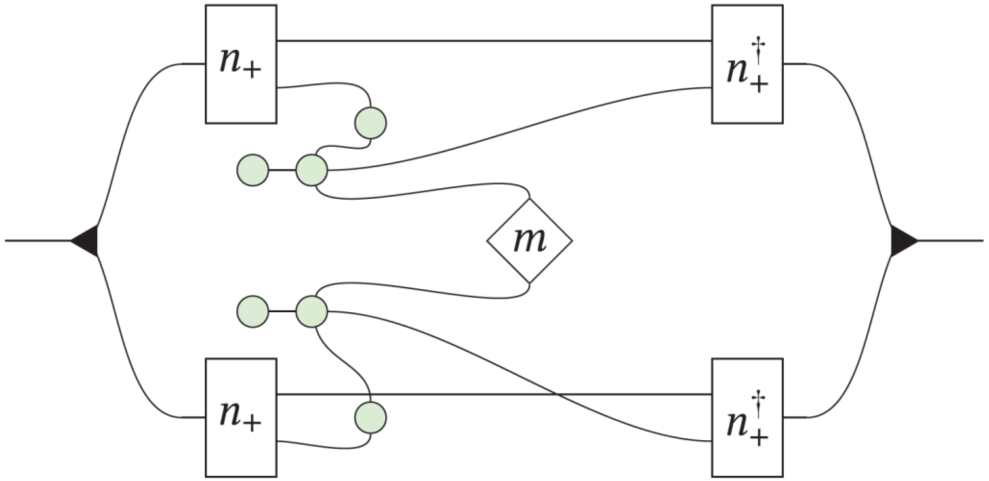

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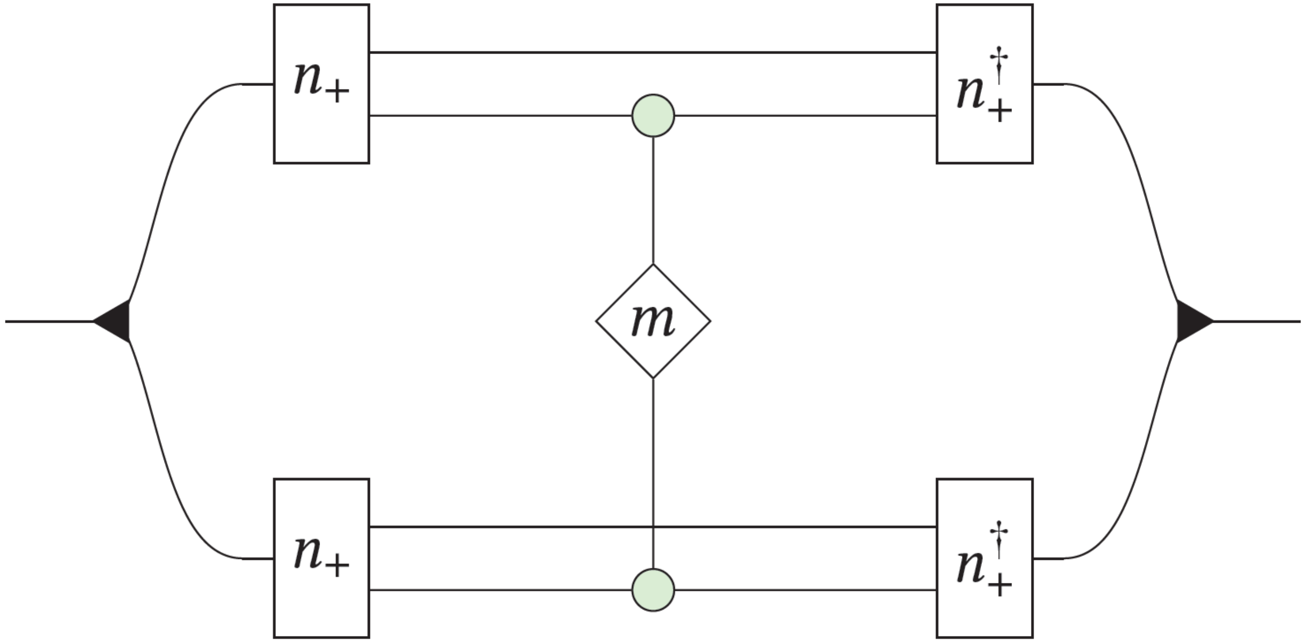

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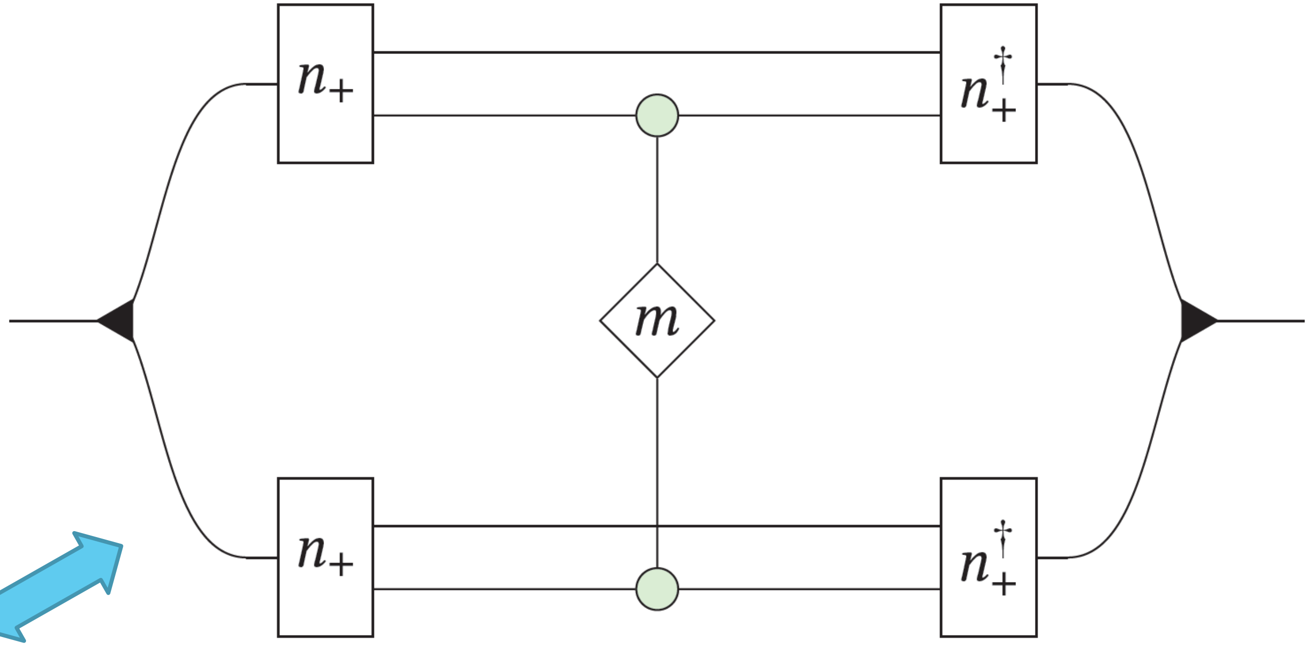

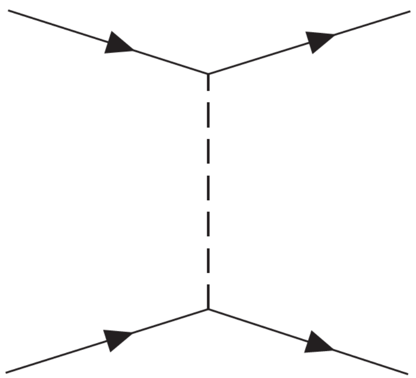

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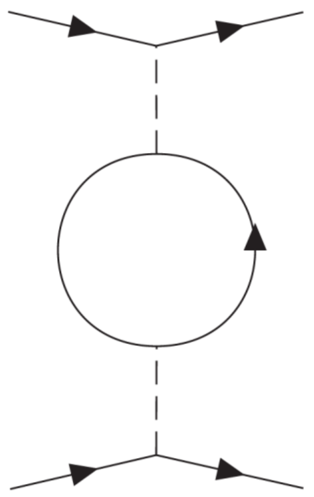

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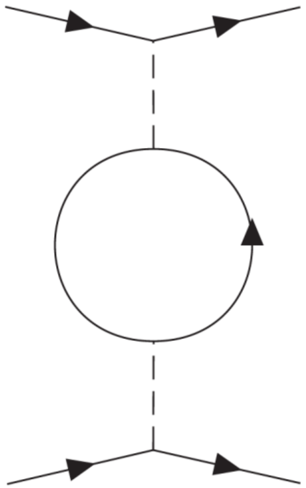

$$\frac{(-ig)^4}{4!}\frac{1}{(\omega_{uv}^4)^4}\sum_{x_1,x_2,x_3,x_4}\begin{pmatrix} n_+^\dagger(x_1)n_+^\dagger(x_2)n_+(x_1)n_+(x_2) \\ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \\ \psi^\dagger(x_3)\psi(x_3)\psi^\dagger(x_4)\psi(x_4) \end{pmatrix}$$

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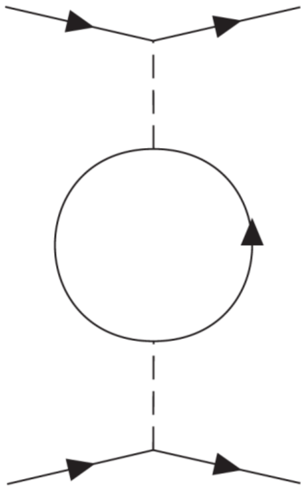

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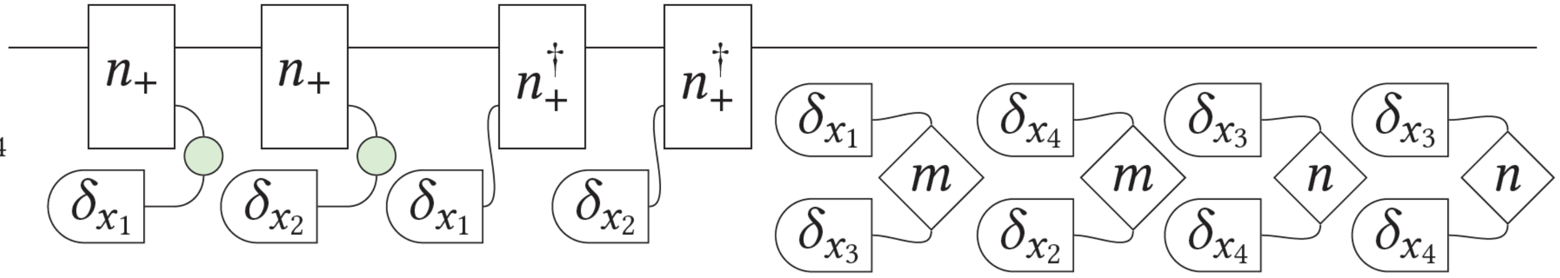

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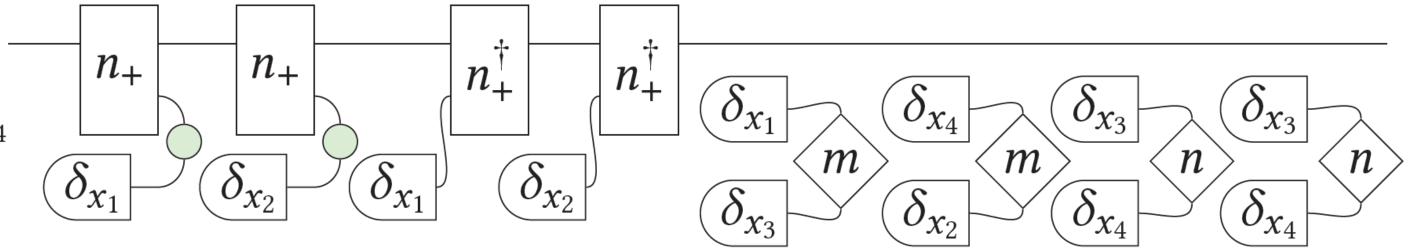

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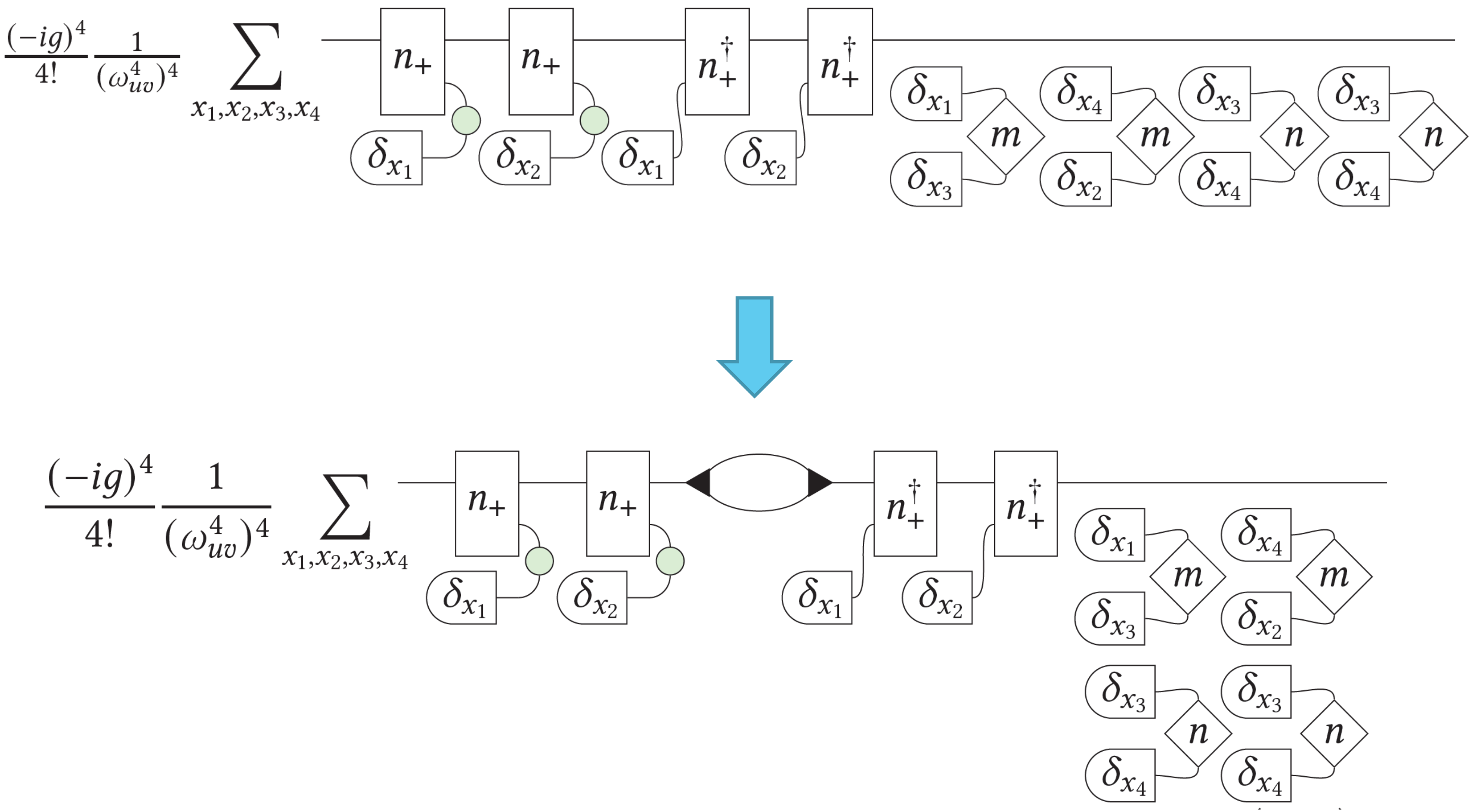

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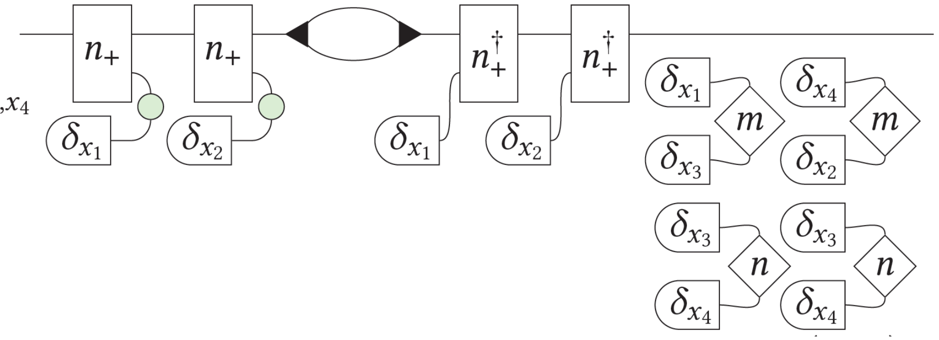

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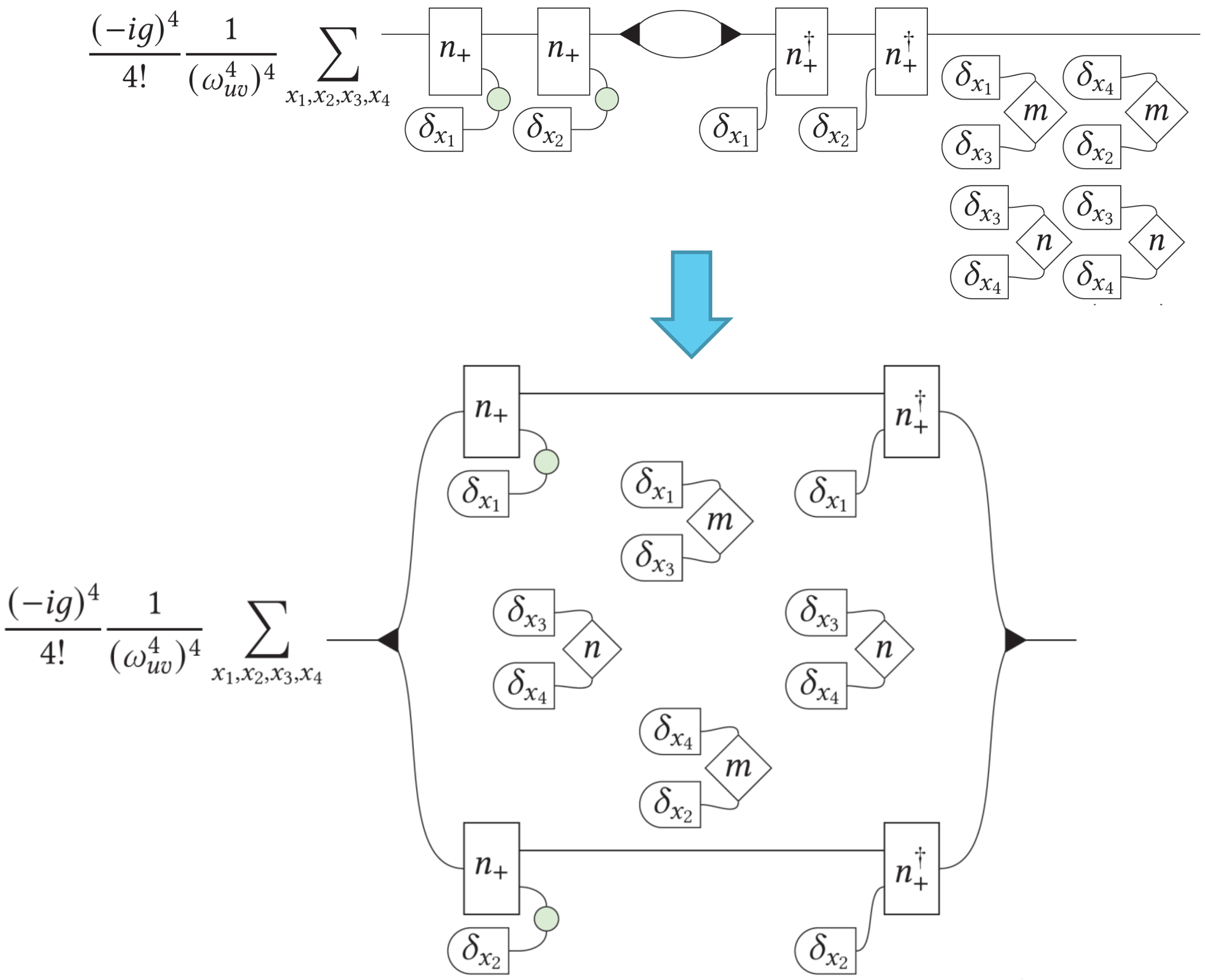

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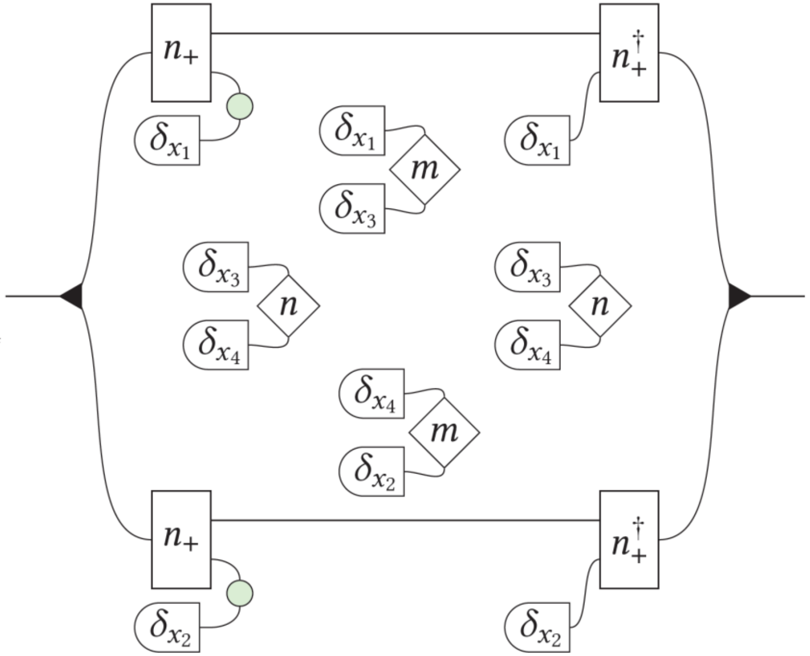

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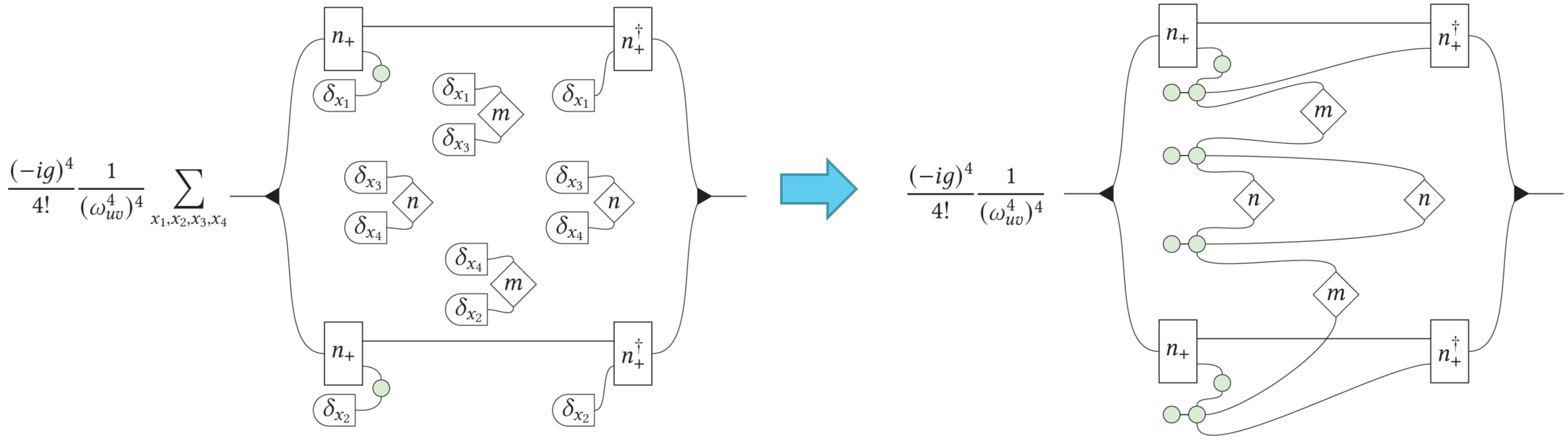

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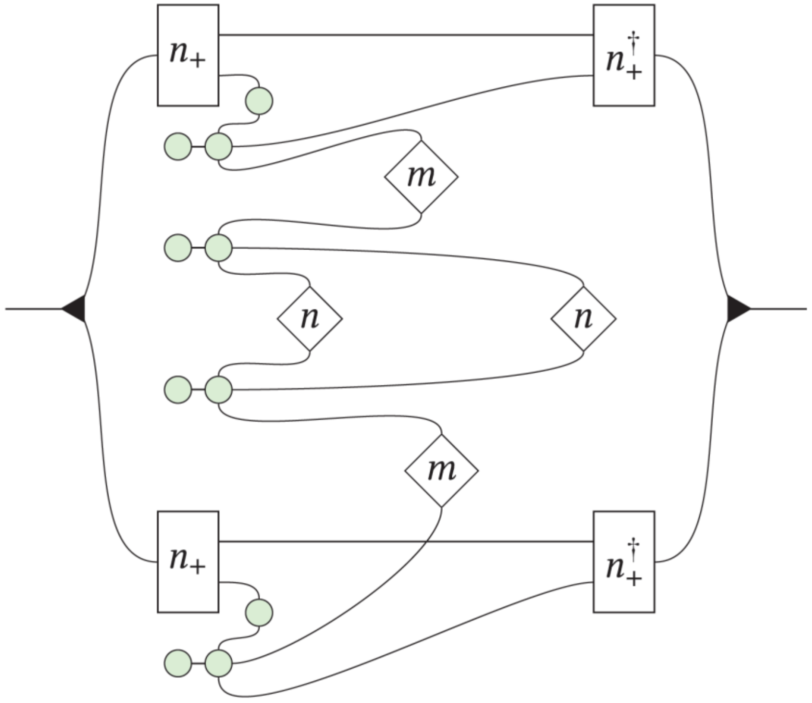

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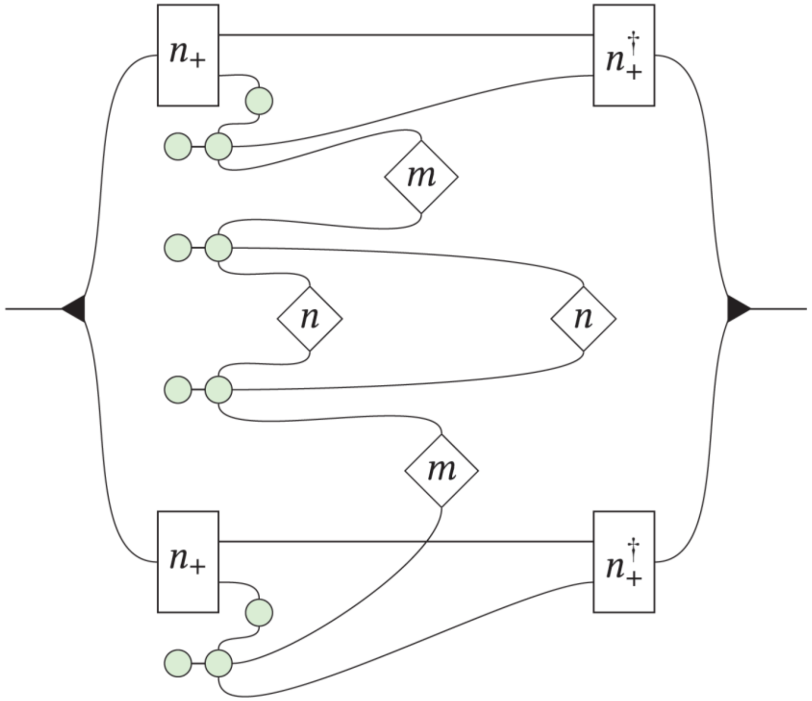

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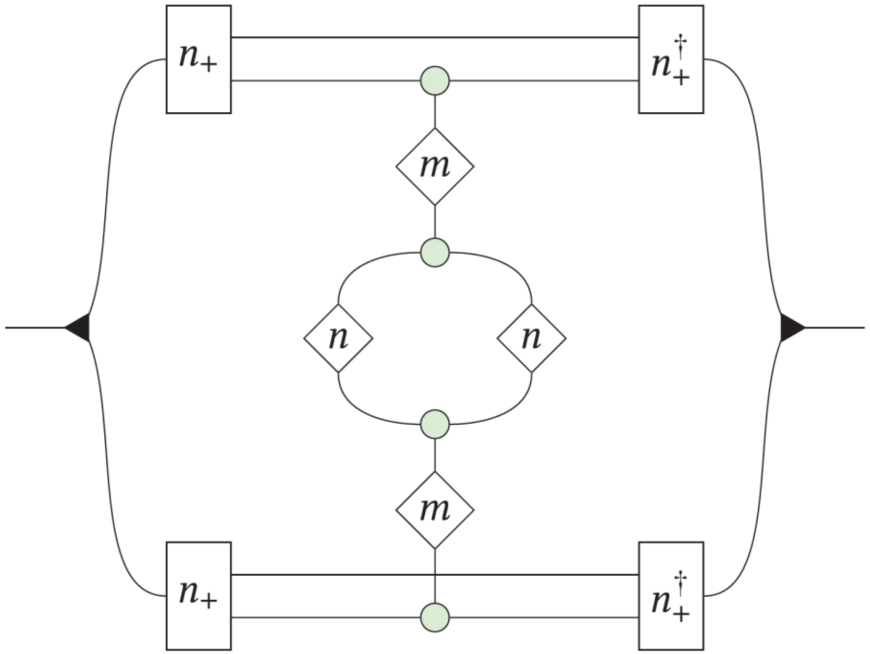

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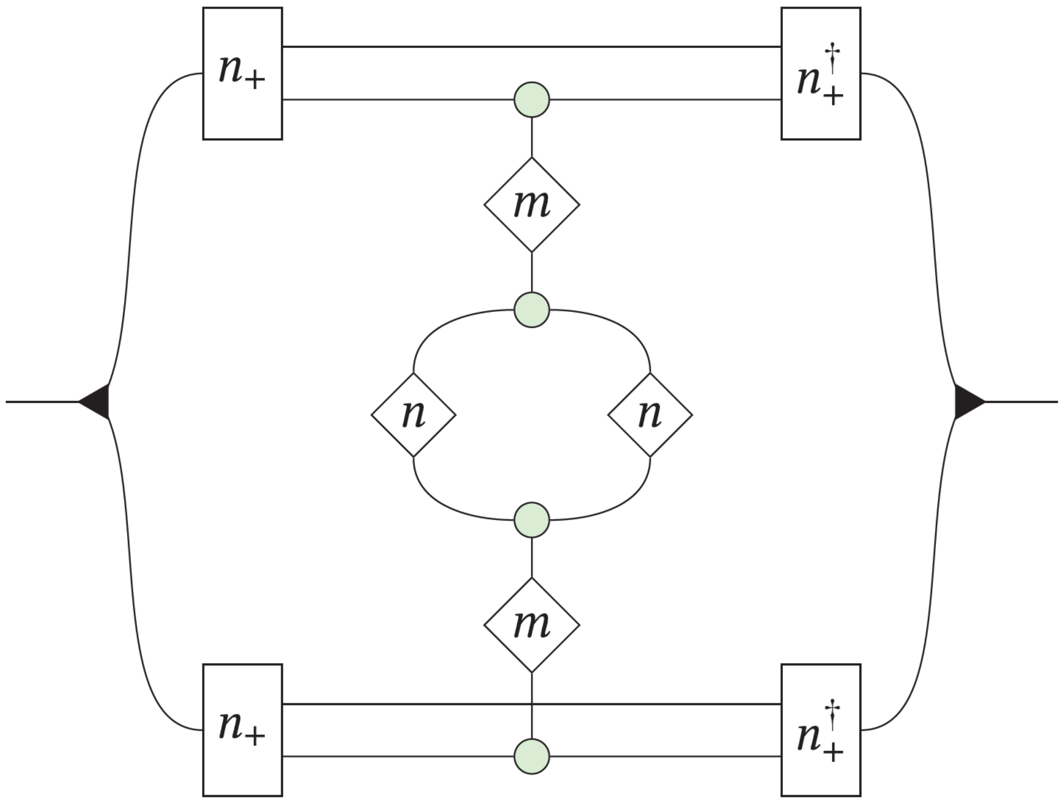

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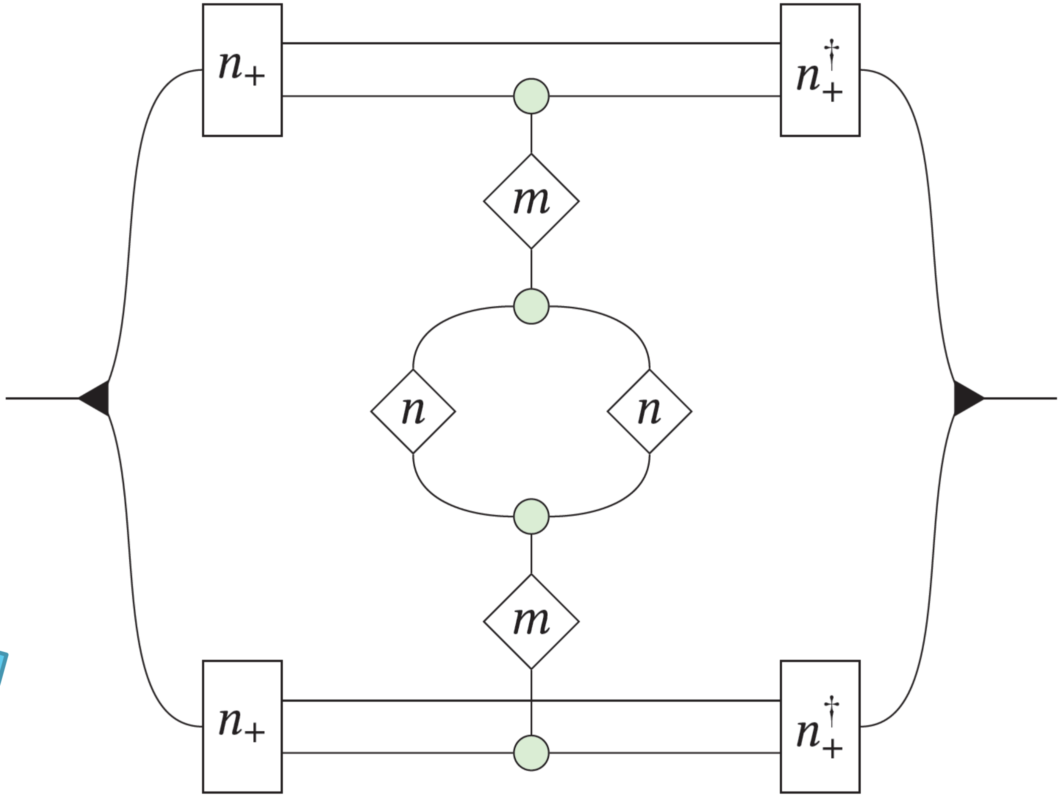
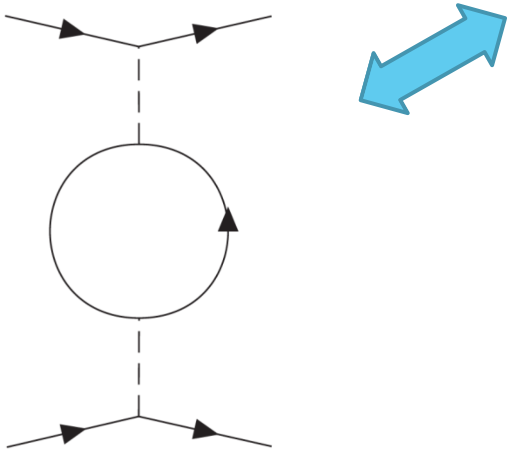
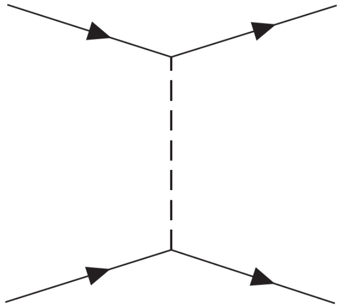
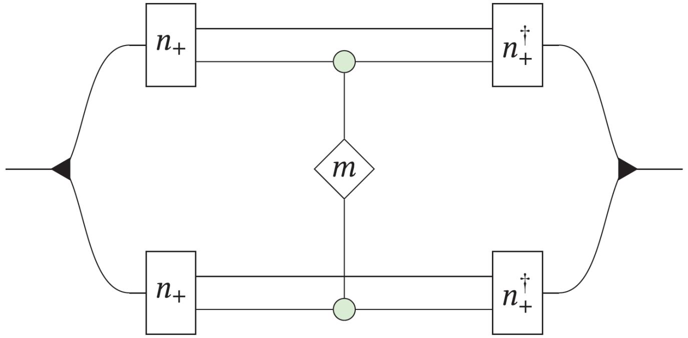


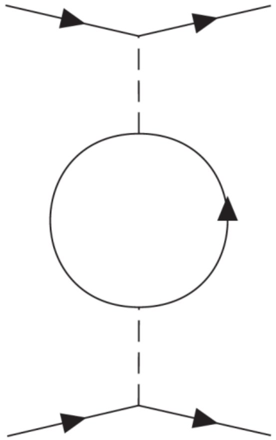


Example Gallery



$$\frac{(-ig)^2}{2}\frac{1}{\omega_{uv}^8}$$





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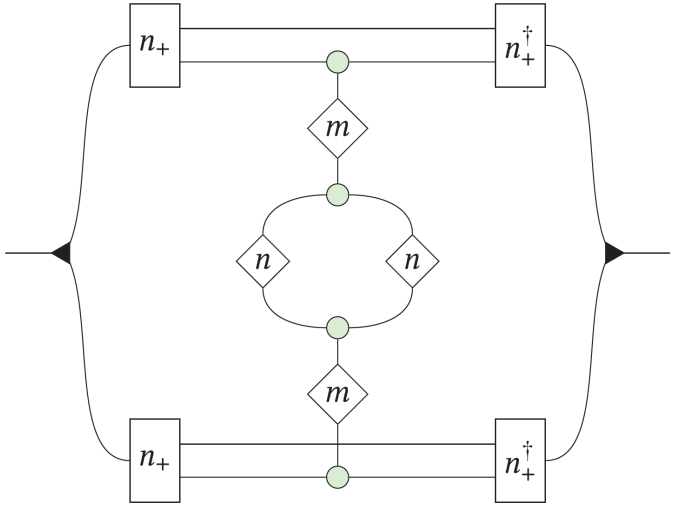

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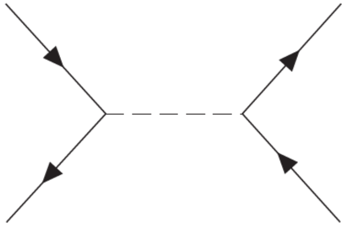

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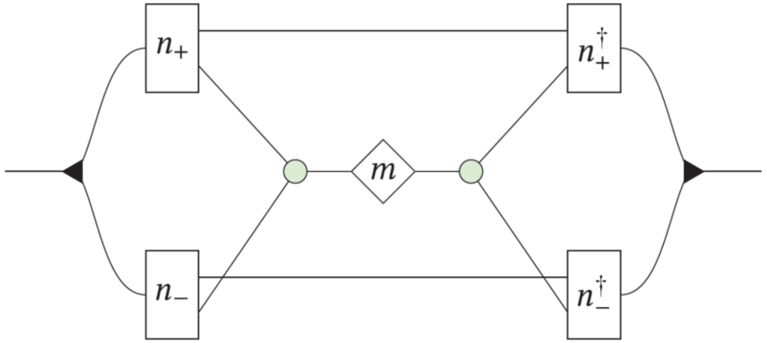
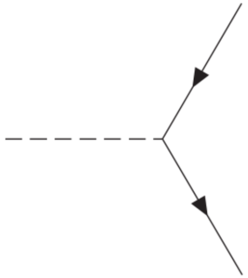
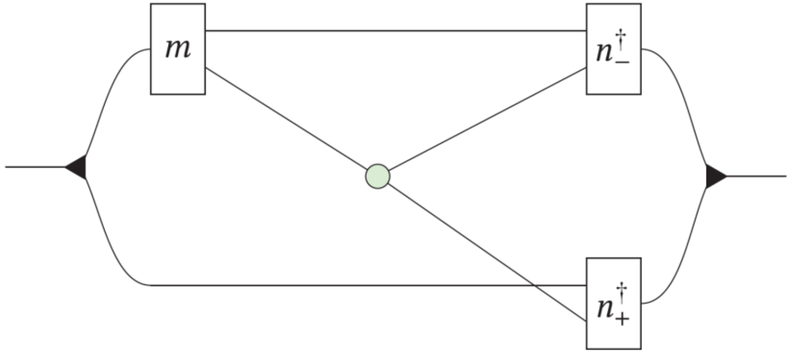
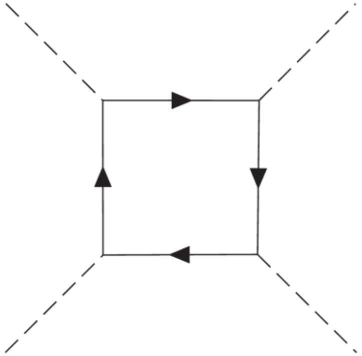




$$(-ig)\frac{1}{\omega_{uv}^4}$$





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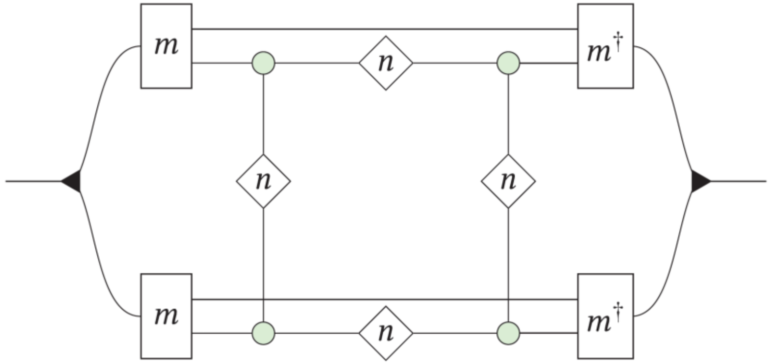
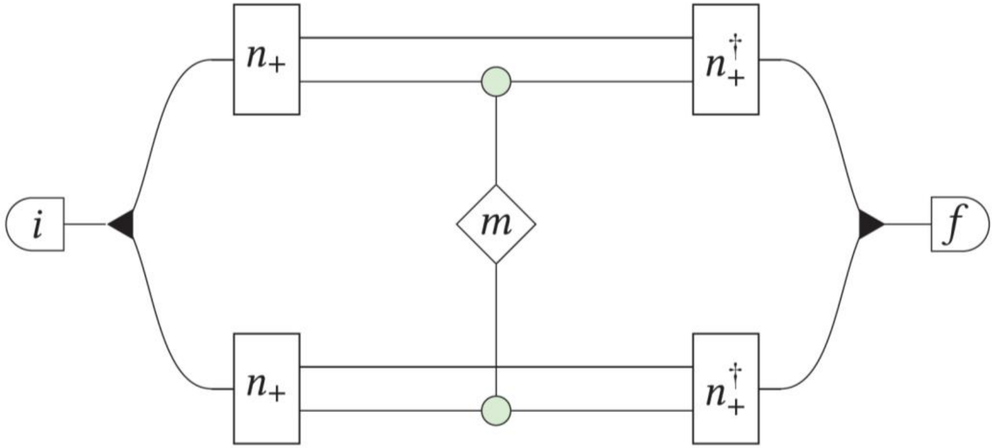


Computing Amplitudes

Computing Amplitudes

$$|i\rangle = |p_1, p_2\rangle = n_+^\dagger(p_1)n_+^\dagger(p_2)\,|0\rangle$$

$$|f\rangle = |p_1', p_2'\rangle = n_+^\dagger(p_1')n_+^\dagger(p_2')\,|0\rangle$$



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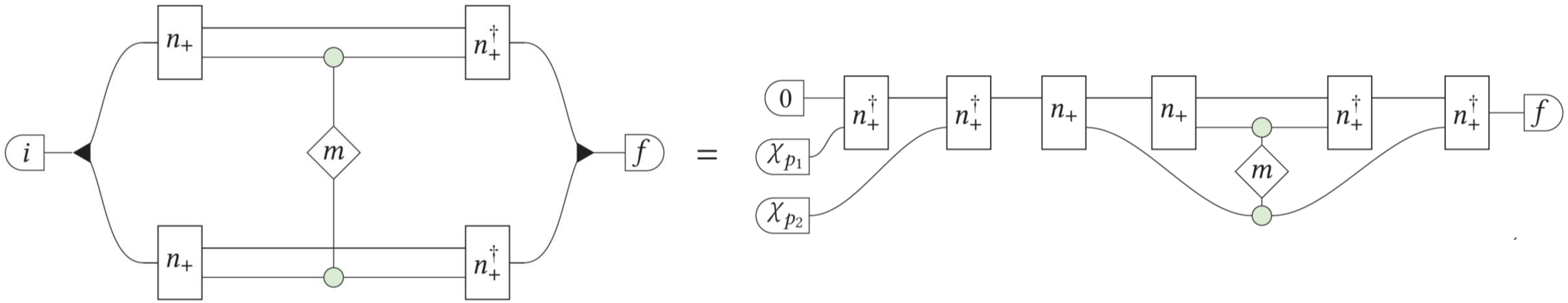

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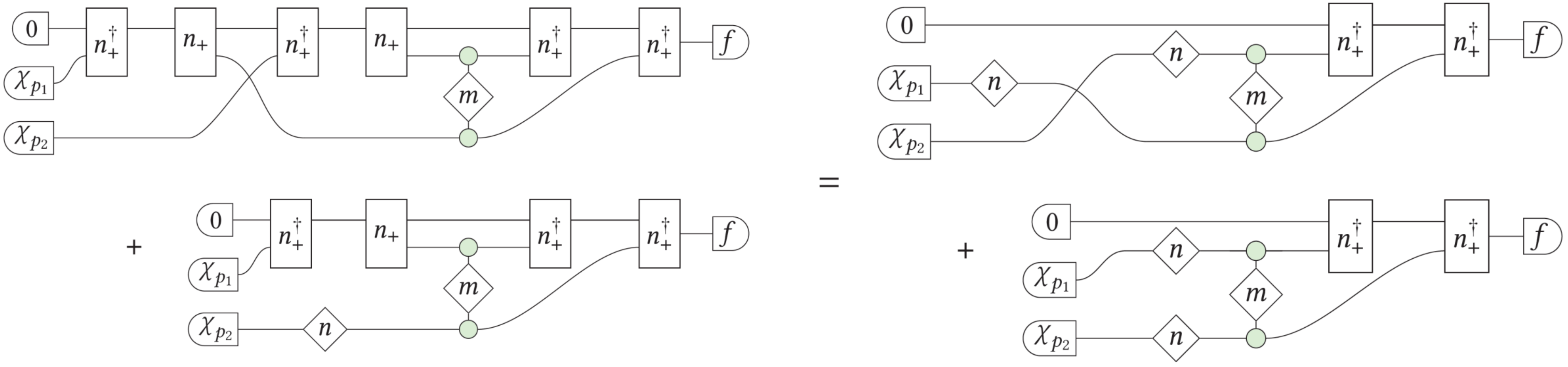
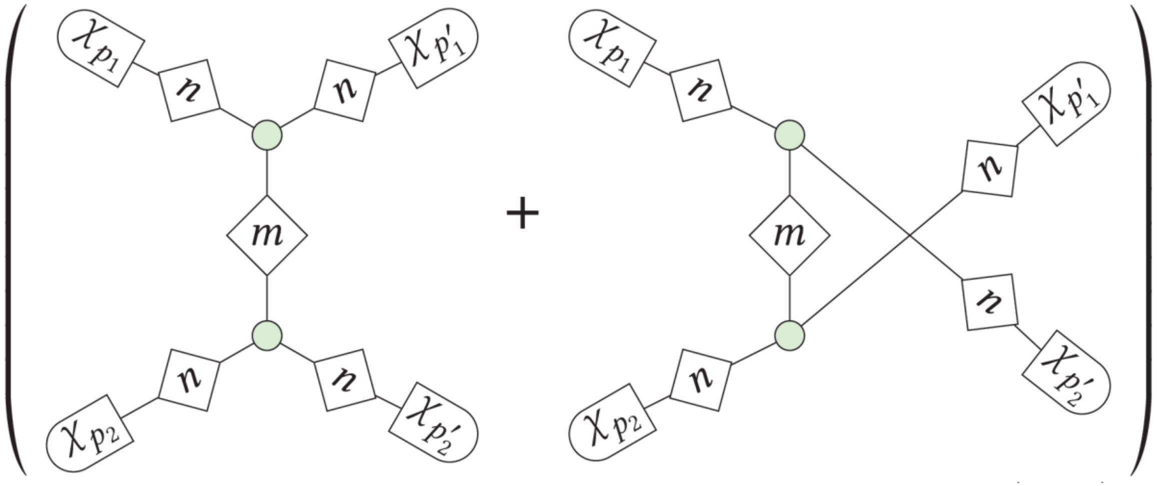
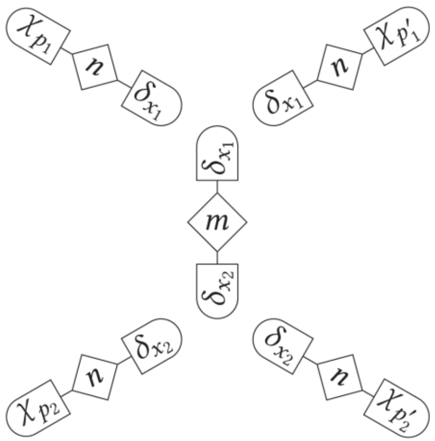


Computing Amplitudes

$(-ig)^2 \frac{1}{\omega_{uv}^8}$



$(-ig)^2 \frac{1}{\omega_{uv}^8} \sum_{x_1,x_2}$



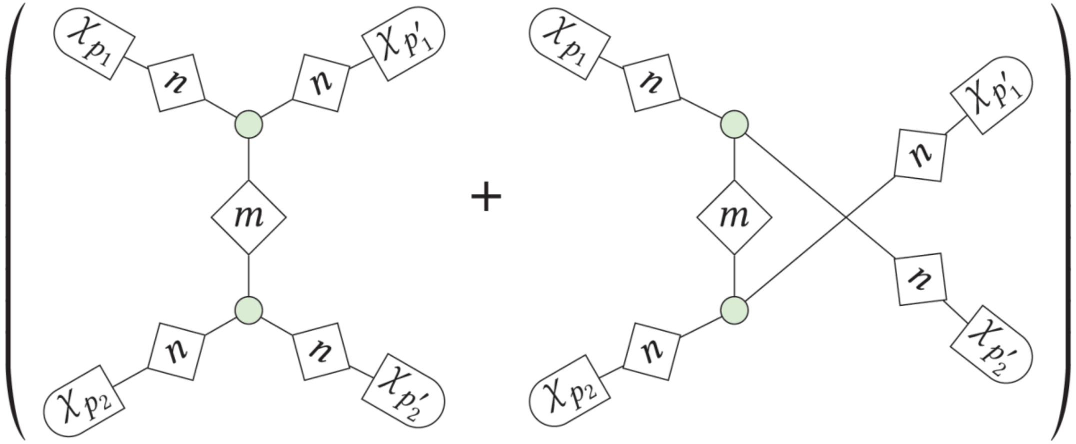
$= e^{i2\pi p\cdot x}$

$$(-ig)^2 \sum_{x_1,x_2} \frac{1}{\omega_{uv}^8} e^{-i2\pi p_1\cdot x_1} e^{-i2\pi p_2\cdot x_2} e^{i2\pi p'_1\cdot x_1} e^{i2\pi p'_2\cdot x_2}$$

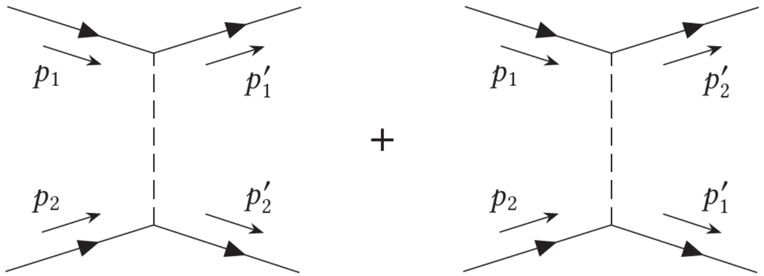
$$\sum_k \frac{1}{\omega_{ir}^4} \frac{ie^{-i2\pi k\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon}$$

Computing Amplitudes

$$(-ig)^2 \frac{1}{\omega_{uv}^8}$$



$$i\mathcal{A} = (-ig)^2 \left[\frac{i}{(p_1 - p'_1)^2 - m^2 + i\epsilon} + \frac{i}{(p_1 - p'_2)^2 - m^2 + i\epsilon} \right]$$

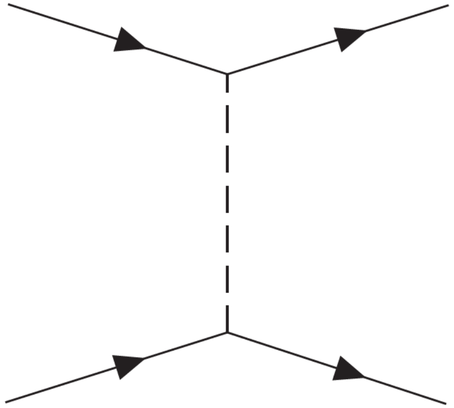


Overview

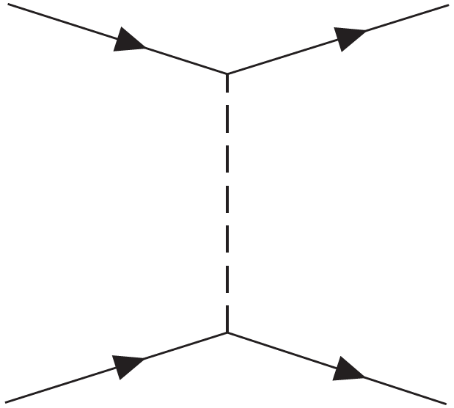
- Categorical quantum fields
- Categorical Feynman diagrams
- **Composing Feynman diagrams**

Composing Feynman Diagrams

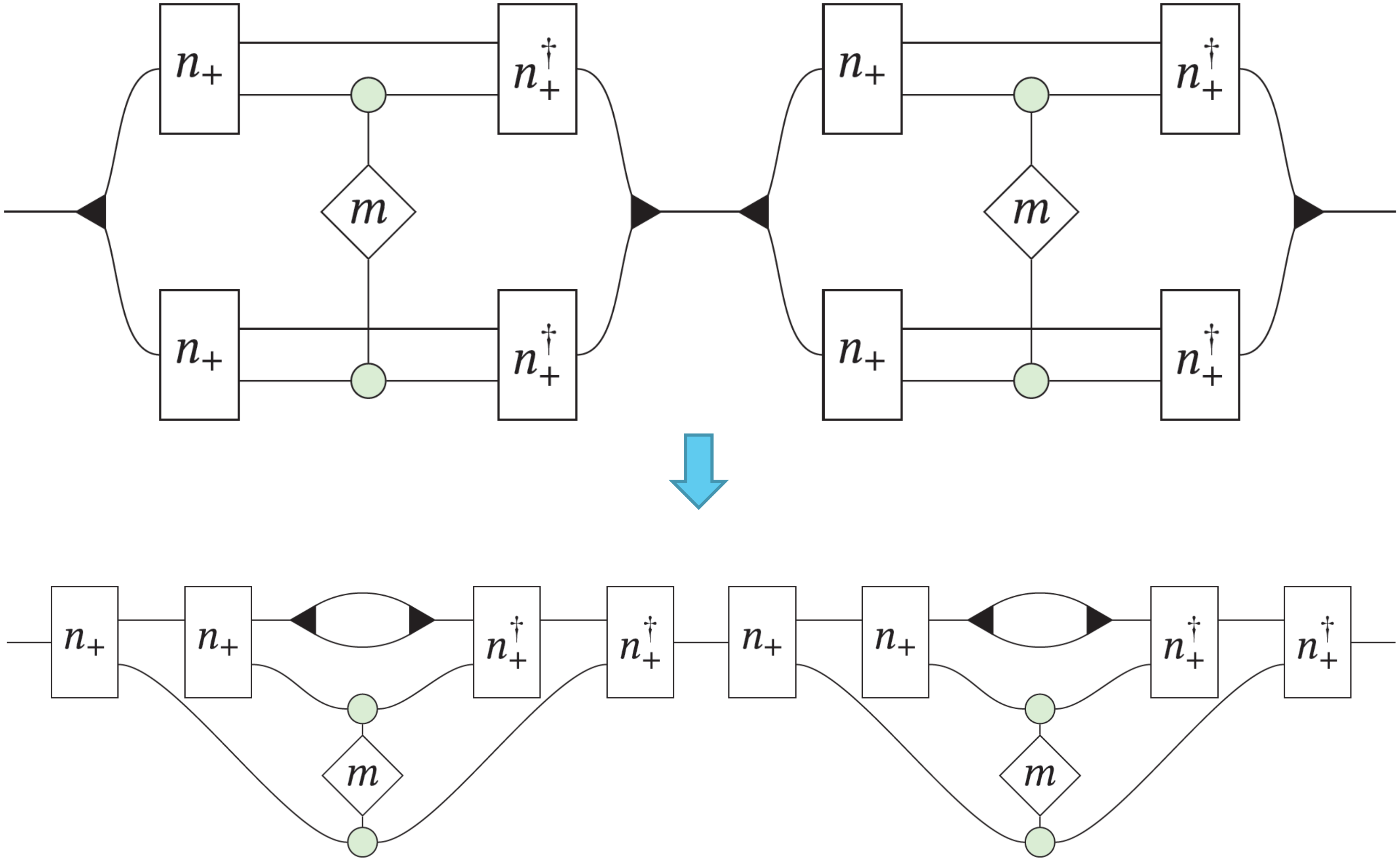
We compose categorical versions of



and



Composing Feynman Diagrams



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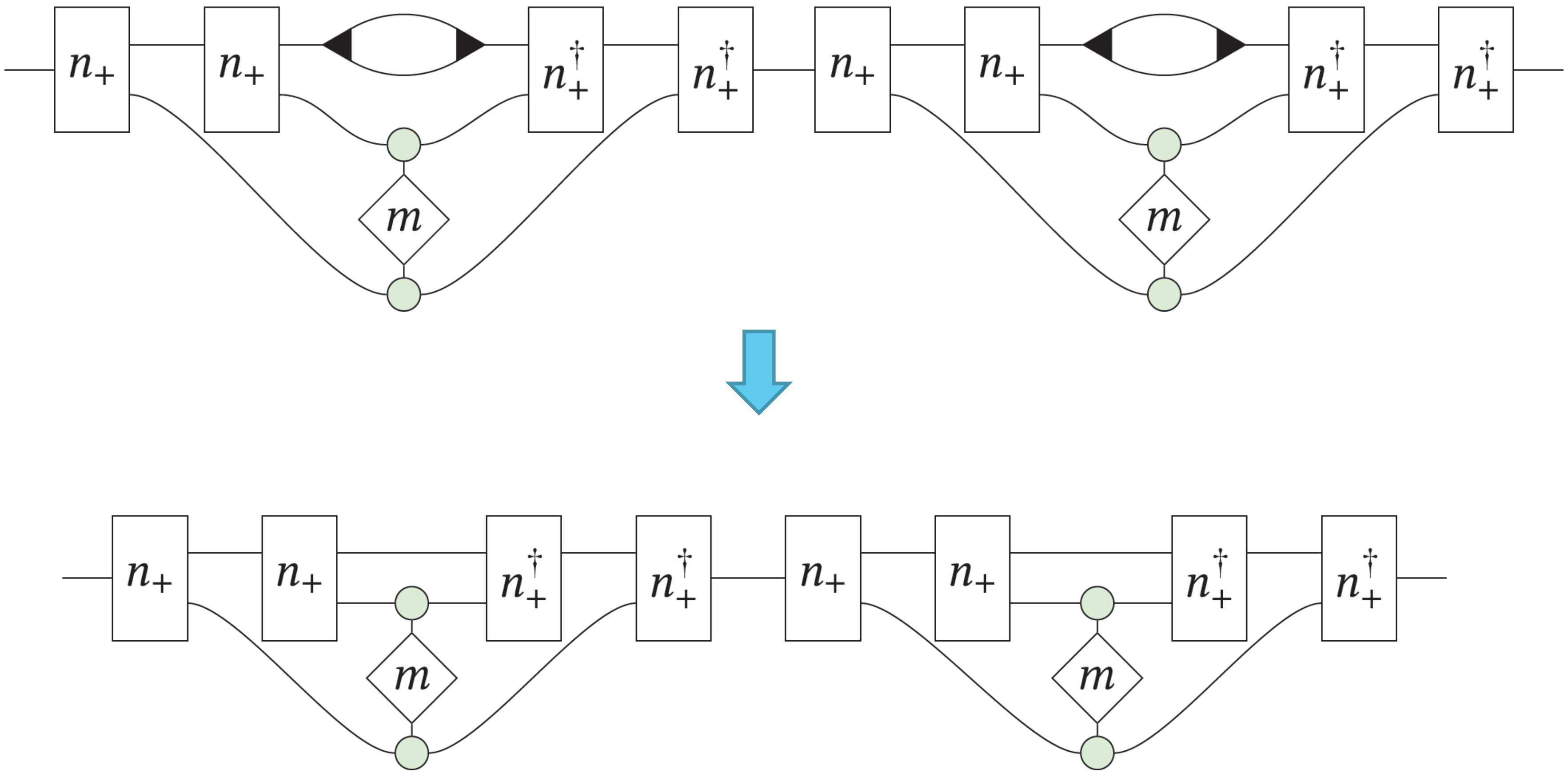

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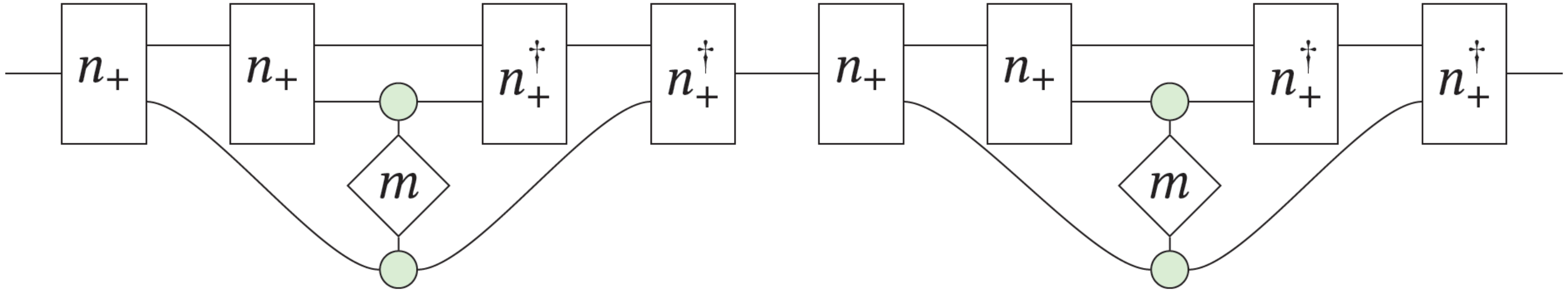

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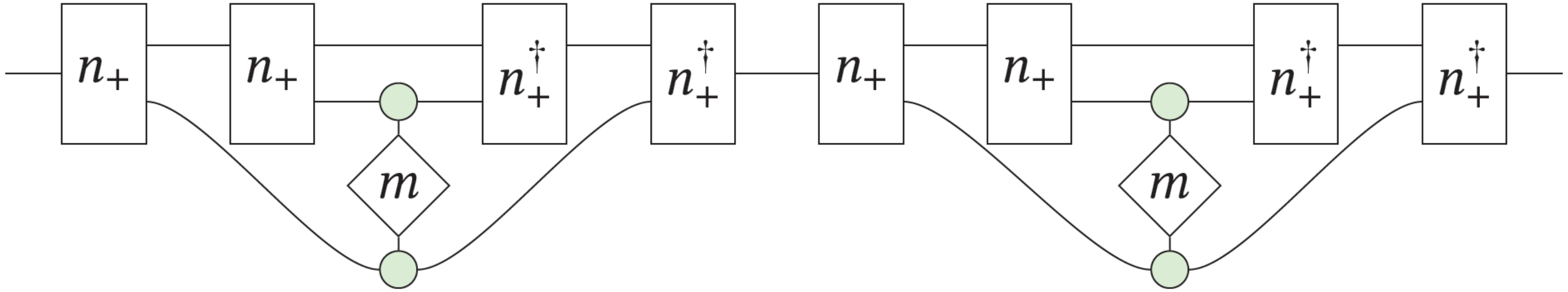
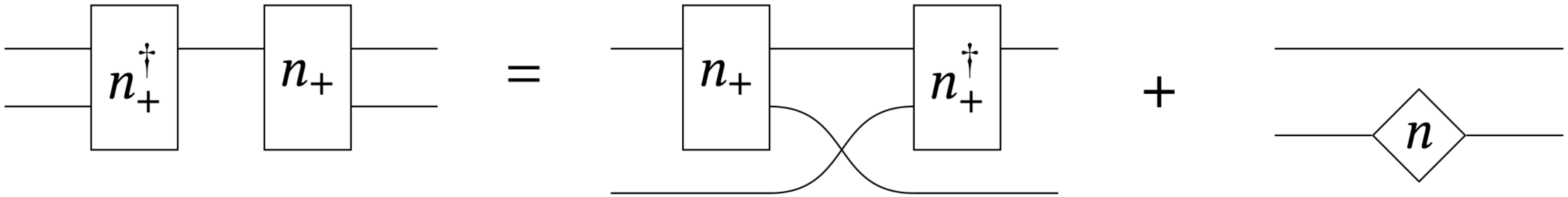


Apply commutation rule:



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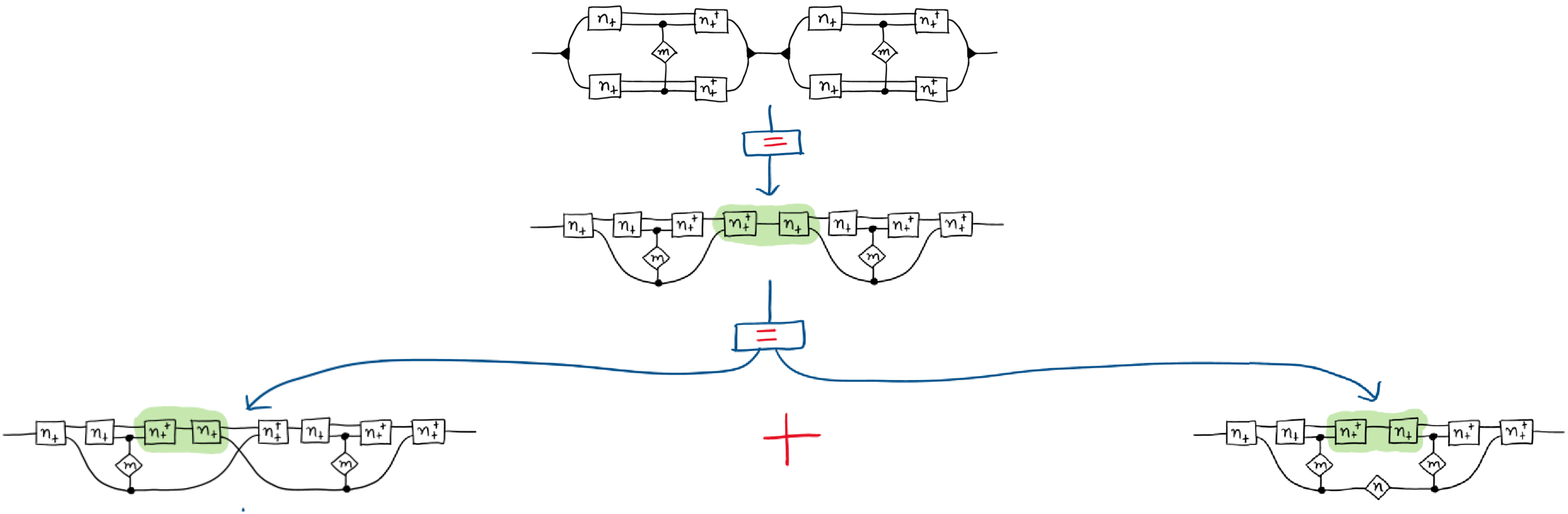

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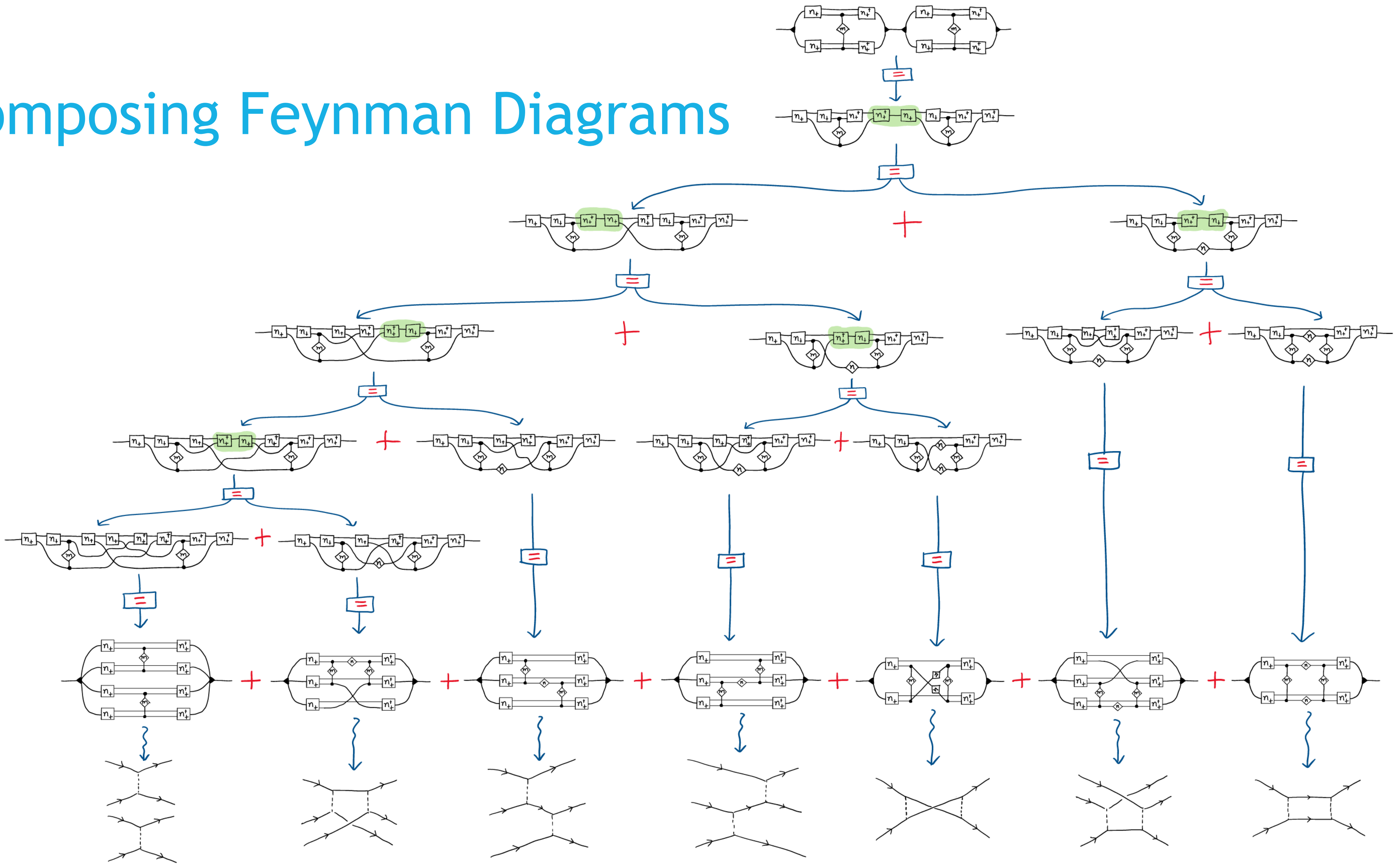

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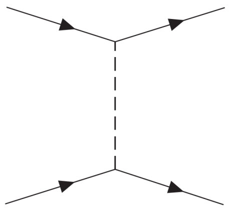

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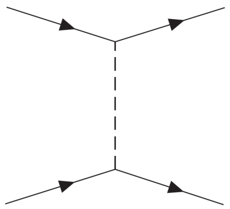

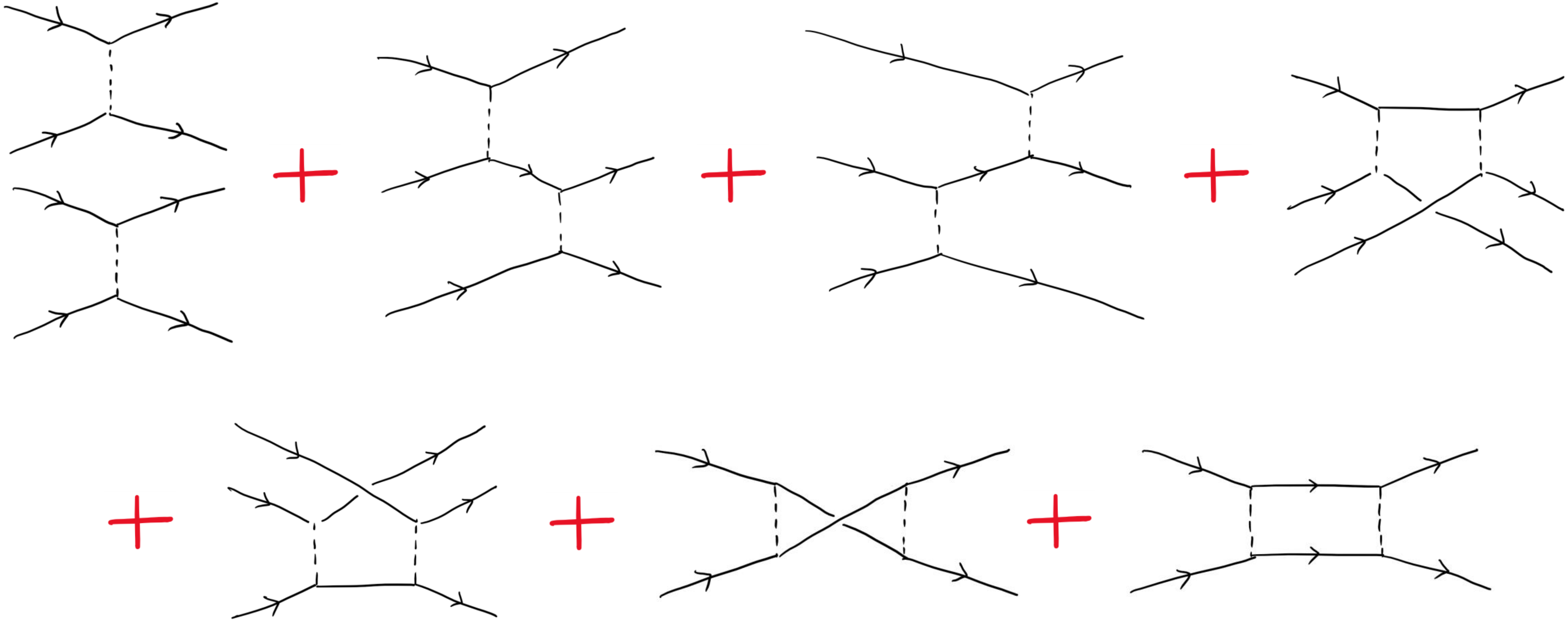

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Conclusion

- Categorical Feynman diagrams represent interaction processes instead of amplitudes
- Amplitudes can be obtained by plugging in initial and final states
- Shift from *syntactic*, graph-theoretic compositionality
  to *semantic*, categorical-diagrammatic compositionality
- Composition of categorical diagrams gives the superposition of all graph-theoretic combinations

Future work

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- Quantum simulation of particle physics
  - Compile categorical Feynman diagrams to quantum circuits

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Thanks for listening!