A Probability Monad on Measure Spaces

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- If $|X| \geq 2$, $\mathcal{D}(X)$ is infinite, so it has to be defined on **Set**. Then **FinStoch** $\hookrightarrow \mathcal{K}\ell(\mathcal{D})$ is the full subcategory on finite sets.
- We cannot handle probabilities such as sequences of independent coin flips on $2^{\mathbb{N}}$ or Lebesgue measure on $[0,1]$ this way. We need a different category to play the role of **Set**.

Compact Hausdorff spaces and C*-algebras

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- But the important part is $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ is an equivalence, where morphisms in $\mathbf{CC^*Alg}^{\mathrm{op}}$ are unital *-homomorphisms. (Gel'fand Duality).

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- $\mathrm{Spec} : \mathbf{CC^*Alg}^{\mathrm{op}} \to \mathbf{CHaus}$ is the inverse where $\mathrm{Spec}(A) = \mathbf{CC^*Alg}(A, \mathbb{C})$.

Positive Unital Maps

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- $\mathbf{CC^*Alg}_{\mathrm{PU}}$ has positive unital maps as morphisms, $\mathbf{CC^*Alg}$ is a subcategory.
- The state space $\mathcal{S}(A) = \mathbf{CC^*Alg}_{\mathrm{PU}}(A, \mathbb{C})$.

The Radon Monad

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- $\mathcal{K}\ell(\mathcal{R})$ is like $\mathcal{K}\ell(\mathcal{D})$ but with continuity.

- We can extend C to a functor $C_{PU} : \mathcal{K}\ell(\mathcal{R}) \to \mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}}$.

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- On $f : X \to \mathcal{R}(Y)$ we define $C_{PU}(f) : C(Y) \to C(X)$ by

$$C(f)(b)(x) = f(x)(b)$$

i.e. swapping the arguments of a curried function.

Probabilistic Gel'fand Duality

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  i.e. swapping the arguments of a curried function.
- C_{PU} is an equivalence. [FJ15]

Probabilistic Gel'fand Duality II

$$\begin{array}{ccc}
\mathcal{K}\ell(\mathcal{R}) & \xrightarrow{C_{PU}} & \mathbf{CC^*Alg}_{\mathrm{PU}}^{\mathrm{op}} \\
F_{\mathcal{R}} \uparrow \dashv \downarrow G_{\mathcal{R}} & & \hookuparrow \dashv \downarrow C \circ \mathcal{S} \\
\mathbf{CHaus} & \xrightarrow[C]{} & \mathbf{CC^*Alg}^{\mathrm{op}},
\end{array}$$

$\mathbf{CC^*Alg}_{\mathrm{PU}}$ is therefore the coKleisli category of a comonad on $\mathbf{CC^*Alg}$.

Given a finite set X and $\phi \in \mathcal{D}(X)$, and a function $\mathcal{Y} : X \to Y$ we can define a function $e : Y \to \mathcal{D}(X)$

$$e(y)(x) = \mathbb{P}(\mathcal{X} = x \mid \mathcal{Y} = y) = \frac{\mathbb{P}(\mathcal{X} = x, \mathcal{Y} = y)}{\mathbb{P}(\mathcal{Y} = y)}$$

$$= \frac{\phi(x)[\mathcal{Y}(x) = y]}{\sum_{x' \in \mathcal{Y}^{-1}(y)} \phi(x')}$$

(where $\mathcal{X} : X \to X$ is the identity function)

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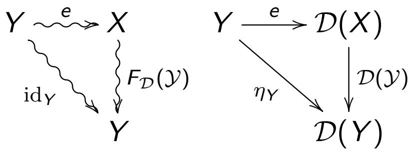

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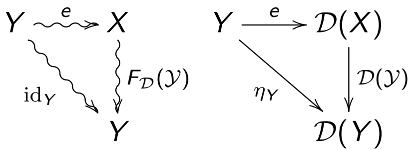

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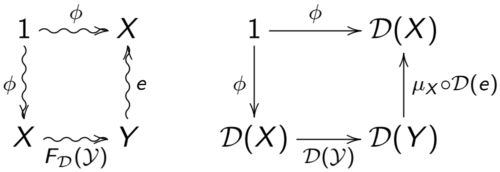
2. ϕ is mapped back to itself by the maps the other way



(marginal probability and conditional probability reproduce joint probability)

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Idea

How about working in a category of measure spaces that ignores null sets to begin with?

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- Under probabilistic Gel'fand duality, a conditional probability map corresponds to the notion of a *conditional expectation* from operator algebra [Tom57, Tak72].
- This is not a coincidence (but no Kleisli categories were used in defining it originally).
- We need the measure theoretic analogue of C, which is L^∞.

L^∞ the W*-algebra

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- The pairing $\langle \text{-}, \text{-} \rangle : L^\infty(X, \nu) \times L^1(X, \nu) \to \mathbb{C}$ defined by integration

  $$\langle a, \phi \rangle = \int_X a\phi \, \mathrm{d}\nu$$

  defines an isometry $L^\infty(X, \nu) \to L^1(X, \nu)^*$. This makes $L^\infty(X, \nu)$ a commutative *W*-algebra*, $L^1(X, \nu)$ is the *predual*.

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  defines an isometry $L^\infty(X, \nu) \to L^1(X, \nu)^*$. This makes $L^\infty(X, \nu)$ a commutative *W*-algebra*, $L^1(X, \nu)$ is the *predual*.
- In fact we cannot stay confined to probability spaces, but we cannot be too general because $L^\infty(X, \nu) \not\cong L^1(X, \nu)$ for all measure spaces.

Gel'fand Duality for W*-algebras

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- L^∞ : **Meas** $\to$ **CW*Alg**$^{\mathrm{op}}$ is an equivalence.

- An inverse to L^∞ is given by $\mathrm{Spec} : \mathbf{CW^*Alg}^{\mathrm{op}} \to \mathbf{Meas}$ (hyperstonean spaces).

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for some family of cardinals $(\kappa_i)_{i \in I}$ (Maharam's theorem).

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- Reference for W*-algebra Gel'fand duality: [Pav22].

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- We want a monad T on **Meas** whose Kleisli category is equivalent to ${\mathbf{CW^*Alg}_{\mathrm{PU}}}^{\mathrm{op}}$. We can use W*-Gel'fand duality to work on the W*-side first.
- So show that $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ has a left adjoint F such that the comparison functor for the coKleisli category of the comonad is an equivalence.

Double Duals

- The forgetful functor $\mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ has a left adjoint, the *enveloping W*-algebra*. For $A \in \mathbf{CC^*Alg}$ it is the double dual A^{**}. This also produces a left adjoint to $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$.

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- Observe:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) &\cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) \cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B)\\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B).\end{aligned}$$

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- It must be that $F(A^{**}) = C(\mathcal{S}(A))^{**}$.
- Not all W*-algebras are double duals!

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- This coequalizer is preserved by the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$.
- Since left adjoints preserve colimits and $\mathbf{CW^*Alg}$ is cocomplete, this allows us to define $F : \mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CW^*Alg}$.

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- Since left adjoints preserve colimits and **CW*Alg** is cocomplete, this allows us to define $F : \mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CW^*Alg}$.
- The coKleisli comparison functor is an equivalence with $\mathbf{CW^*Alg}_{\mathrm{PU}}$ because $\mathbf{CW^*Alg}_{\mathrm{PU}}$ and **CW*Alg** have the same objects. [Wes17, Theorem 9]

Main Result

Theorem

There is a monad T on **Meas** *such that* $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg}_{\mathrm{PU}}$.

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- It seems the simplest way to realize $T(X)$ is to take the Gel'fand spectrum of $F(L^\infty(X))$.

Final Remarks

- For a countable set X

$$T(X) \cong ([0,1], \mathcal{P}([0,1]), \nu_d) + ([0,1]^2, \mathcal{P}([0,1]) \otimes \widehat{\mathcal{B}o([0,1])}, \nu_d \otimes \nu_L)$$

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  where ν_d is the counting measure and ν_L the Lebesgue measure.

- The need to use non-probabilistic spaces is analogous to the need to use **Set** instead of **FinSet** to define $\mathcal{D}$.

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  $$T(X) \cong ([0,1], \mathcal{P}([0,1]), \nu_d) + ([0,1]^2, \mathcal{P}([0,1]) \otimes \widehat{\mathcal{B}o([0,1])}, \nu_d \otimes \nu_L)$$

  where ν_d is the counting measure and ν_L the Lebesgue measure.
- The need to use non-probabilistic spaces is analogous to the need to use **Set** instead of **FinSet** to define $\mathcal{D}$.
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- It should be that $\mathcal{K}\ell(T)$ and $\mathbf{CW^*Alg}_{\mathrm{PU}}{}^{\mathrm{op}}$ are Markov categories in the sense of [Fri20] (work in progress).

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