Diagrammatic Presentations of Enriched Monads and Theories for a Subcategory of Arities

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$\mathcal{C}$ " " " $\mathcal{V}$-category

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$$\mathcal{J} \hookrightarrow \mathcal{C} \quad \underbrace{\text{subcategory of arities}}_{\text{dense full sub-}\mathcal{V}\text{-category}}$$

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$\mathcal{C}$ " " " $\mathcal{V}$-category

$\mathcal{J} \hookrightarrow \mathcal{C}$ $\underbrace{\text{subcategory of arities}}_{\text{dense full sub-}\mathcal{V}\text{-category}}$

$\mathcal{J}$ small

A parametrized ($\mathcal{J}$-ary) operation on an object A of $\mathcal{C}$ is a morphism

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$$\omega : \mathcal{C}(J, A) \longrightarrow \mathcal{C}(C, A) \quad \text{in } \mathcal{V}$$

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for objects $J \in \text{ob}\,\mathcal{J}$ (the arity) and $C \in \text{ob}\,\mathcal{C}$ (the parameter) .

A parametrized ($\mathscr{J}$-ary) operation on an object A of $\mathscr{C}$ is a morphism

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(Equivalently, $\omega : \mathscr{C}(J,A) \otimes C \longrightarrow A$ in $\mathscr{C}$.)

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(Equivalently, $\omega : \mathcal{C}(J,A) \otimes C \longrightarrow A$ in $\mathcal{C}$.)

E.g. $\mathcal{J} := \mathrm{FinCard} \hookrightarrow \mathcal{C} := \mathcal{V} := \mathrm{Set}$

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$J = \overset{\omega}{n}$

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(Equivalently, $\omega : \mathcal{C}(J,A) \otimes C \longrightarrow A$ in $\mathcal{C}$.)

E.g. $\mathcal{J} := \mathrm{FinCard} \hookrightarrow \mathcal{C} := \mathcal{V} := \mathrm{Set}$

$J = \overset{\omega}{n}$ $\qquad\qquad C, A^{\epsilon}$

A parametrized ($\mathcal{J}$-ary) operation on an object A of $\mathcal{C}$ is a morphism

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$C \in \mathrm{ob}\,\mathcal{C}$ (the parameter).

(Equivalently, $\omega : \mathcal{C}(J,A) \otimes C \longrightarrow A$ in $\mathcal{C}$.)

E.g. $\mathcal{J} := \mathrm{FinCard} \hookrightarrow \mathcal{C} := \mathcal{V} := \mathrm{Set}$

$J = \overset{\in}{n}$ $\qquad\qquad\qquad C, A^{\in}$

$\mathcal{C}(J,A) = \mathrm{Set}(n,A) = A^n$.

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(Equivalently, $\omega : \mathcal{C}(J,A) \otimes C \longrightarrow A$ in $\mathcal{C}$.)

E.g. $\mathcal{J} := \mathrm{FinCard} \hookrightarrow \mathcal{C} := \mathcal{V} := \mathrm{Set}$

$J = \overset{\in}{n}$ $\qquad$ $C, \overset{\in}{A}$

$\mathcal{C}(J,A) = \mathrm{Set}(n,A) = A^n.$ $\qquad$ $\omega : A^n \times C \longrightarrow A$ in Set

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	Kind of algebraic structure A	Parametrized operations $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	Arities J	Parameters C
cartesian closed category					

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$				

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$			

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$		

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$	2	

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$	2	1

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$	2	1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	Kind of algebraic structure A	Parametrized operations $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	Arities J	Parameters C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$	2 0	1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$	2 0	1 1

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\overset{\cong}{A \times R}}{R \times A} \longrightarrow A$	2 0	1 1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\wr\wr \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\bullet : \underset{\underset{A \times R}{\cong}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set					

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\cong}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)				

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\shortparallel}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A			

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$		

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$	

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cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\cong}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathbf{Gph}_{fp} \hookrightarrow \mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2],A) \to \mathbf{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$	$[1] = (\bullet \to \bullet)$

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in V, for an internal rig R in V	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\underset{A\times R}{\cong}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$	$[1] = (\bullet \to \bullet)$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\overset{\cong}{A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\cdot \to \cdot \to \cdot)$	$[1] = (\cdot \to \cdot)$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$ $0 : A^0 \to A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \to A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{array}{c} R \times A \\ \wr\| \\ A \times R \end{array} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathbf{Gph}_{fp} \hookrightarrow \mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2],A) \to \mathbf{Gph}([1],A)$ $e : \mathbf{Gph}([0],A) \longrightarrow \mathbf{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\cong}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\cdot \to \cdot \to \cdot)$ $[0] = (\cdot)$	$[1] = (\cdot \to \cdot)$ $[1]$
Cat					

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$				

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\overset{R \times A}{\cong}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A			

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{matrix} R\times A \\ \cong \\ A\times R \end{matrix} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathsf{Gph}_{fp} \hookrightarrow \mathsf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathsf{Gph}([2],A) \to \mathsf{Gph}([1],A)$ $e : \mathsf{Gph}([0],A) \to \mathsf{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathsf{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$		

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\wr \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$	2	

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in V, for an internal rig R in V	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{array}{c} R \times A \\ \cong \\ A \times R \end{array} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$	2	1

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in V, for an internal rig R in V	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\underset{A \times R}{\wr\Vert}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$	2	1

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in V, for an internal rig R in V	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\bullet : \underset{\underset{A \times R}{\cong}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2], A) \to \mathrm{Gph}([1], A)$ $e : \mathrm{Gph}([0], A) \to \mathrm{Gph}([1], A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$	2 0	1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\overset{\cong}{A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$	2 0	1 1

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	*Internal left R-module A in V, for an internal rig R in V*	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\wr\|}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\cdot \to \cdot \to \cdot)$ $[0] = (\cdot)$	$[1] = (\cdot \to \cdot)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{2}}$	2 0	1 1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$ $0 : A^0 \to A$ $\cdot : \underset{A \times R}{\overset{R \times A}{\cong}} \to A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\cdot \to \cdot \to \cdot)$ $[0] = (\cdot)$	$[1] = (\cdot \to \cdot)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \to A$ $i : A^0 \to A$ $\alpha : A^3 \to A^{\tilde{2}}$	2 0 3	1 1

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	Kind of algebraic structure A	Parametrized operations $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	Arities J	Parameters C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	*Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$*	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\wr\|}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{\mathrm{fp}} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathbb{I}}}$	2 0 3	1 1 $\tilde{\mathbb{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$

V	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow V$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in V, for an internal rig R in V	$+ : A^2 \to A$ $0 : A^0 \to A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \to A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \to A$ $i : A^0 \to A$ $\alpha : A^3 \to A^{\tilde{\mathbb{2}}}$ $\lambda, \rho : A \to A^{\tilde{\mathbb{2}}}$	2 0 3	1 1 $\tilde{\mathbb{2}} = (\bullet \underset{\sim}{\rightleftarrows} \bullet)$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{matrix} R \times A \\ \cong \\ A \times R \end{matrix} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{2}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{2}}$	2 0 3 1	1 1 $\tilde{2} = (\bullet \underset{\sim}{\rightleftarrows} \bullet)$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\underset{A \times R}{\rotatebox{90}{$\cong$}}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathfrak{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathfrak{I}}}$	2 0 3 1	1 1 $\tilde{\mathfrak{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathfrak{I}}$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	*Internal left R-module A in V, for an internal rig R in V*	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\bullet : \underset{A \times R}{\underset{\cong}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathfrak{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathfrak{I}}}$	2 0 3 1	1 1 $\tilde{\mathfrak{I}} = (\bullet \underset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathfrak{I}}$
cartesian closed category					

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{array}{c} R \times A \\ \cong \\ A \times R \end{array} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.				

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\overset{\text{\rotatebox{90}{$\cong$}}}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]			

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\underset{\cong}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathfrak{2}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathfrak{2}}}$	2 0 3 1	1 1 $\tilde{\mathfrak{2}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathfrak{2}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \to A^L$ ("lookup")		

$\mathcal{V}$	$\mathcal{J}\hookrightarrow\mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A)\to\mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A)\otimes C\to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n\in\mathbb{N}\}\hookrightarrow\mathcal{V}$ where we write $n := \coprod_{i\in n} 1$ $(n\in\mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2\longrightarrow A$ $0 : A^0\longrightarrow A$ $\cdot : \begin{matrix} R\times A \\ \wr\Vert \\ A\times R\end{matrix}\longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp}\hookrightarrow\mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A)\to\mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A)\longrightarrow\mathrm{Gph}([1],A)$	$[2]=(\bullet\to\bullet\to\bullet)$ $[0]=(\bullet)$	$[1]=(\bullet\to\bullet)$ $[1]$
Cat	$\mathbb{N}\hookrightarrow\mathrm{Cat}$	Small monoidal category A	$\otimes : A^2\longrightarrow A$ $i : A^0\longrightarrow A$ $\alpha : A^3\longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda,\rho : A\longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}}=(\bullet\overset{\sim}{\rightleftarrows}\bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J}\hookrightarrow\mathcal{V}$ closed under × with $1\in\mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V\longrightarrow A^L$ ("lookup")	$V\in\mathrm{ob}\,\mathcal{J}$ ("values")	

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{matrix} R \times A \\ \cong \\ A \times R \end{matrix} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2], A) \to \mathrm{Gph}([1], A)$ $e : \mathrm{Gph}([0], A) \to \mathrm{Gph}([1], A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values")	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations")

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\overset{R \times A}{\cong}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values")	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations")

$\mathscr{V}$	$\mathscr{J} \hookrightarrow \mathscr{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathscr{C}(J,A) \to \mathscr{C}(C,A)$ or $\omega : \mathscr{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathscr{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathscr{V}$, for an internal rig R in $\mathscr{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{matrix} R \times A \\ \wr\shortparallel \\ A \times R \end{matrix} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathcal{I}}}$	2 0 3 1	1 1 $\tilde{\mathcal{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathcal{I}}$
cartesian closed category	Any small $\mathscr{J} \hookrightarrow \mathscr{V}$ closed under $\times$ with $1 \in \mathscr{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\mathscr{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations")

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{\overset{\wr\wr}{R \times A}} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2], A) \to \mathrm{Gph}([1], A)$ $e : \mathrm{Gph}([0], A) \to \mathrm{Gph}([1], A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathbb{2}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathbb{2}}}$	2 0 3 1	1 1 $\tilde{\mathbb{2}} = (\bullet \overset{\sim}{\longleftrightarrow} \bullet)$ $\tilde{\mathbb{2}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** **A**	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** **J**	**Parameters** **C**
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\substack{\cong \\ A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{\mathrm{fp}} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathbb{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathbb{I}}}$	2 0 3 1	1 1 $\tilde{\mathbb{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathbb{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
〃					

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\underset{A \times R}{\wr\wr}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \longrightarrow \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\widetilde{\mathbb{I}}}$ $\lambda, \rho : A \longrightarrow A^{\widetilde{\mathbb{I}}}$	2 0 3 1	1 1 $\widetilde{\mathbb{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\widetilde{\mathbb{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
''	with $\begin{tikzcd} & \overset{''}{\mathcal{J}} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$				

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$ $0 : A^0 \to A$ $\cdot : \underset{A \times R}{\overset{\shortparallel}{R \times A}} \to A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{\mathrm{fp}} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\cdot \to \cdot \to \cdot)$ $[0] = (\cdot)$	$[1] = (\cdot \to \cdot)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \to A$ $i : A^0 \to A$ $\alpha : A^3 \to A^{\tilde{\mathbb{I}}}$ $\lambda, \rho : A \to A^{\tilde{\mathbb{I}}}$	2 0 3 1	1 1 $\tilde{\mathbb{I}} = (\cdot \overset{\sim}{\rightleftarrows} \cdot)$ $\tilde{\mathbb{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \to A^L$ ("lookup") $u : A \to A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
''	'' with $$\begin{tikzcd} & \mathcal{J} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$$	Model A of the parametrized theory of instantiating and reading bits [Staton]			

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{array}{c} R \times A \\ \rotatebox{90}{$\cong$} \\ A \times R \end{array} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{\mathrm{fp}} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathfrak{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathfrak{I}}}$	2 0 3 1	1 1 $\tilde{\mathfrak{I}} = (\bullet \underset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathfrak{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \to A^L$ ("lookup") $u : A \to A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
〃	with $\mathbb{N} \hookrightarrow \overset{〃}{\mathcal{J}} \hookrightarrow \mathcal{V}$, $\mathbb{N} \hookrightarrow \mathcal{V}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2 \times J \longrightarrow A$		

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \ (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \begin{array}{c} R \times A \\ \cong \\ A \times R \end{array} \longrightarrow A$	2 0 1	1 1 R
$\mathbf{Set}$	$\mathbf{Gph}_{fp} \hookrightarrow \mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2], A) \to \mathbf{Gph}([1], A)$ $e : \mathbf{Gph}([0], A) \to \mathbf{Gph}([1], A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
$\mathbf{Cat}$	$\mathbb{N} \hookrightarrow \mathbf{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathfrak{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathfrak{I}}}$	2 0 3 1	1 1 $\tilde{\mathfrak{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathfrak{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \to A^L$ ("lookup") $u : A \to A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
″	″ with $\begin{tikzcd} & \mathcal{J} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2 \times J \longrightarrow A$	2	

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1$ $(n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{\overset{\shortparallel}{A \times R}}{R \times A} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathbf{Gph}_{fp} \hookrightarrow \mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2],A) \to \mathbf{Gph}([1],A)$ $e : \mathbf{Gph}([0],A) \to \mathbf{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathbf{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\tilde{\mathbb{I}}}$ $\lambda, \rho : A \longrightarrow A^{\tilde{\mathbb{I}}}$	2 0 3 1	1 1 $\tilde{\mathbb{I}} = (\bullet \overset{\sim}{\rightleftarrows} \bullet)$ $\tilde{\mathbb{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under × with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
"	with $\begin{tikzcd} & \overset{''}{\mathcal{J}} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2 \times J \longrightarrow A$	2	$J \in \mathrm{ob}\,\mathcal{J}$ ("locations")

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \; (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \to A$ $0 : A^0 \to A$ $\bullet : \begin{smallmatrix} R \times A \\ \cong \\ A \times R \end{smallmatrix} \longrightarrow A$	2 0 1	1 1 R
Set	$\mathrm{Gph}_{fp} \hookrightarrow \mathrm{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathrm{Gph}([2],A) \to \mathrm{Gph}([1],A)$ $e : \mathrm{Gph}([0],A) \to \mathrm{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
Cat	$\mathbb{N} \hookrightarrow \mathrm{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\widetilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\widetilde{\mathcal{I}}}$	2 0 3 1	1 1 $\widetilde{\mathcal{I}} = (\bullet \underset{\sim}{\rightleftarrows} \bullet)$ $\widetilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \to A^L$ ("lookup") $u : A \to A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
"	with $\begin{tikzcd} & \overset{''}{\mathcal{J}} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2 \times J \longrightarrow A$ $v : A^J \longrightarrow A$	2	$J \in \mathrm{ob}\,\mathcal{J}$ ("locations")

$\mathcal{V}$	$\mathcal{J} \hookrightarrow \mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A) \to \mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A) \otimes C \to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$ where we write $n := \coprod_{i \in n} 1 \quad (n \in \mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2 \longrightarrow A$ $0 : A^0 \longrightarrow A$ $\cdot : \underset{A \times R}{R \times A} \overset{\wr\wr}{} \longrightarrow A$	2 0 1	1 1 R
$\mathbf{Set}$	$\mathbf{Gph}_{fp} \hookrightarrow \mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2],A) \to \mathbf{Gph}([1],A)$ $e : \mathbf{Gph}([0],A) \longrightarrow \mathbf{Gph}([1],A)$	$[2] = (\bullet \to \bullet \to \bullet)$ $[0] = (\bullet)$	$[1] = (\bullet \to \bullet)$ $[1]$
$\mathbf{Cat}$	$\mathbb{N} \hookrightarrow \mathbf{Cat}$	Small monoidal category A	$\otimes : A^2 \longrightarrow A$ $i : A^0 \longrightarrow A$ $\alpha : A^3 \longrightarrow A^{\widetilde{\mathcal{I}}}$ $\lambda, \rho : A \longrightarrow A^{\widetilde{\mathcal{I}}}$	2 0 3 1	1 1 $\widetilde{\mathcal{I}} = (\bullet \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \bullet)$ $\widetilde{\mathcal{I}}$
cartesian closed category	Any small $\mathcal{J} \hookrightarrow \mathcal{V}$ closed under $\times$ with $1 \in \mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V \longrightarrow A^L$ ("lookup") $u : A \longrightarrow A^{L \times V}$ ("update")	$V \in \mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda \in \Lambda} 1$ ("locations") $L \times V$
"	" with $\begin{tikzcd} & \mathcal{J} \arrow[dr] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2 \times J \longrightarrow A$ $\nu : A^J \longrightarrow A$	2 J	$J \in \mathrm{ob}\,\mathcal{J}$ ("locations")

$\mathcal{V}$	$\mathcal{J}\hookrightarrow\mathcal{C}$	**Kind of algebraic structure** A	**Parametrized operations** $\omega : \mathcal{C}(J,A)\to\mathcal{C}(C,A)$ or $\omega : \mathcal{C}(J,A)\otimes C\to A$	**Arities** J	**Parameters** C
cartesian closed category	$\{n \mid n\in\mathbb{N}\}\hookrightarrow\mathcal{V}$ where we write $n := \coprod_{i\in n} 1 \quad (n\in\mathbb{N})$	Internal left R-module A in $\mathcal{V}$, for an internal rig R in $\mathcal{V}$	$+ : A^2\to A$ $0 : A^0\to A$ $\cdot : \underset{A\times R}{\underset{\wr\wr}{R\times A}}\to A$	2 0 1	1 1 R
Set	$\mathbf{Gph}_{fp}\hookrightarrow\mathbf{Gph}$ (finite graphs) (graphs)	Small category A	$c : \mathbf{Gph}([2],A)\to\mathbf{Gph}([1],A)$ $e : \mathbf{Gph}([0],A)\to\mathbf{Gph}([1],A)$	$[2]=(\bullet\to\bullet\to\bullet)$ $[0]=(\bullet)$	$[1]=(\bullet\to\bullet)$ $[1]$
Cat	$\mathbb{N}\hookrightarrow\mathbf{Cat}$	Small monoidal category A	$\otimes : A^2\to A$ $i : A^0\to A$ $\alpha : A^3\to A^{\tilde{\mathbb{I}}}$ $\lambda,\rho : A\to A^{\tilde{\mathbb{I}}}$	2 0 3 1	1 1 $\tilde{\mathbb{I}}=(\bullet\overset{\sim}{\rightleftarrows}\bullet)$ $\tilde{\mathbb{I}}$
cartesian closed category	Any small $\mathcal{J}\hookrightarrow\mathcal{V}$ closed under $\times$ with $1\in\mathcal{J}$.	Global state algebra A [Plotkin-Power]	$\ell : A^V\to A^L$ ("lookup") $u : A\to A^{L\times V}$ ("update")	$V\in\mathrm{ob}\,\mathcal{J}$ ("values") 1	$L := \coprod_{\lambda\in\Lambda} 1$ ("locations") $L\times V$
〃	〃 with $\begin{tikzcd} & \mathcal{J} \arrow[dr, hook] & \\ \mathbb{N} \arrow[ur, hook] \arrow[rr, hook] & & \mathcal{V} \end{tikzcd}$	Model A of the parametrized theory of instantiating and reading bits [Staton]	$? : A^2\times J\to A$ $\nu : A^J\to A$	2 J	$J\in\mathrm{ob}\,\mathcal{J}$ ("locations") 1

A *free-form $\mathcal{J}$-signature* is a set Σ (whose elements we call *operation symbols*) equipped with objects

$$J_\sigma \in \mathrm{ob}\,\mathcal{J}, \qquad C_\sigma \in \mathrm{ob}\,\mathcal{C} \qquad (\sigma \in \Sigma).$$

A free-form $\mathcal{J}$-signature is a set Σ (whose elements we call operation symbols) equipped with objects

$$J_\sigma \in \mathrm{ob}\,\mathcal{J}, \quad C_\sigma \in \mathrm{ob}\,\mathcal{C} \quad (\sigma \in \Sigma).$$

An Σ-algebra is an object A of $\mathcal{C}$ equipped with parametrized operations

$$\sigma^A : \mathcal{C}(J_\sigma, A) \longrightarrow \mathcal{C}(C_\sigma, A) \qquad (\sigma \in \Sigma)$$

A free-form $\mathcal{J}$-signature is a set Σ (whose elements we call operation symbols) equipped with objects

$$J_\sigma \in \mathrm{ob}\,\mathcal{J}, \qquad C_\sigma \in \mathrm{ob}\,\mathcal{C} \qquad (\sigma \in \Sigma).$$

An Σ-algebra is an object A of $\mathcal{C}$ equipped with parametrized operations

$$\sigma^A : \mathcal{C}(J_\sigma, A) \longrightarrow \mathcal{C}(C_\sigma, A) \qquad (\sigma \in \Sigma)$$

(or $\sigma^A : \mathcal{C}(J_\sigma, A) \otimes C \longrightarrow A$).

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$$\begin{tikzcd}
\mathcal{S}\text{-Alg} \arrow[d, "U^{\mathcal{S}}"] \\
\mathcal{C}
\end{tikzcd}$$

A *free-form $\mathcal{J}$-signature* is a set $\mathfrak{S}$ (whose elements we call *operation symbols*) equipped with objects

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$$\begin{tikzcd} \mathfrak{S}\text{-Alg} \arrow[d, "U^{\mathfrak{S}}"] \\ \mathcal{C} \end{tikzcd} \in \operatorname{ob}(\mathcal{V}\text{-CAT}/\mathcal{C})$$

A **diagrammatic $\mathcal{J}$-ary equation** $\omega \doteq \nu$ on $\begin{pmatrix}\mathcal{A}\\ \downarrow G\\ \mathcal{C}\end{pmatrix} \in \mathrm{ob}(\mathcal{V}\text{-CAT}/\mathcal{C})$ is a pair of $\mathcal{V}$-natural transformations

$$\begin{tikzcd}\mathcal{C}(J,G-) \arrow[r, shift left, "\omega"] \arrow[r, shift right, "\nu"'] & \mathcal{C}(C,G-)\end{tikzcd}$$

with $J \in \mathrm{ob}\,\mathcal{J}$ and $C \in \mathrm{ob}\,\mathcal{C}$.

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An object A of $\mathcal{A}$ satisfies $\omega \doteq \nu$ if

$$\omega_A = \nu_A \;:\; \mathcal{C}(J, GA) \longrightarrow \mathcal{C}(C, GA).$$

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i.e., a family of parallel pairs

$$\begin{tikzcd} \mathcal{C}(J,A) \arrow[r, shift left, "\omega_A"] \arrow[r, shift right, "\nu_A"'] & \mathcal{C}(C,A) \end{tikzcd} \quad \left(\text{or } \begin{tikzcd} \mathcal{C}(J,A) \otimes C \arrow[r, shift left, "\omega_A"] \arrow[r, shift right, "\nu_A"'] & A \end{tikzcd} \right)$$

$\mathcal{V}$-natural in $A \in \mathcal{S}\text{-Alg}$.

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$$\begin{array}{llll} \Sigma = \{ & +, & 0, & \cdot\ \} \\ \text{Arity:} & 2 & 0 & 1 \\ \text{Parameter:} & 1 & 1 & R \end{array}$$

A left R-module is an Σ-algebra $A = (A, +^A, 0^A, \cdot^A)$ that satisfies several diagrammatic Σ-equations,

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$$\begin{tikzcd}
R \times A^2 \arrow[r, "\Delta \times 1"] \arrow[d, "1 \times +^A"'] & R^2 \times A^2 \arrow[r, "\sim"] & R \times A \times R \times A \arrow[r, "\cdot^A \times \cdot^A"] & A \times A \arrow[d, "+^A"] \\
R \times A \arrow[rrr, "\cdot^A"] & & & A
\end{tikzcd}$$

$(A \in \mathcal{S}\text{-Alg})$.

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A diagrammatic $\mathcal{T}$-presentation $P = (\mathfrak{S}, E)$ consists of

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$\mathcal{J}$-ary variety

Theorem. Suppose (a) <u>OR</u> (b) below.

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<u>(1)</u>. For every free-form $\mathcal{J}$-presentation $P = (\Sigma, E)$ there is a $\mathcal{V}$-monad $\mathbb{T}_P$ on $\mathcal{C}$ (unique up to isomorphism) such that $P\text{-Alg} \cong \mathbb{T}_P\text{-Alg}$ in $\mathcal{V}\text{-CAT}/\mathcal{C}$.

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<u>(2)</u>. Up to isomorphism, the $\mathcal{V}$-monads of the form $\mathbb{T}_P$ are precisely the $\mathcal{J}$-ary / $\mathcal{J}$-nervous $\mathcal{V}$-monads on $\mathcal{C}$.

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<u>(1)</u>. For every free-form $\mathscr{J}$-presentation $P = (\mathcal{S}, E)$ there is a $\mathscr{V}$-monad $\mathbb{T}_P$ on $\mathscr{C}$ (unique up to isomorphism) such that $P\text{-Alg} \cong \mathbb{T}_P\text{-Alg}$ in $\mathscr{V}\text{-CAT}/\mathscr{C}$.

<u>(2)</u>. Up to isomorphism, the $\mathscr{V}$-monads of the form $\mathbb{T}_P$ are precisely the $\mathscr{J}$-ary / $\mathscr{J}$-nervous $\mathscr{V}$-monads on $\mathscr{C}$.

<u>(3)</u>. There is an equivalence $\mathrm{Mnd}_{\mathscr{J}}(\mathscr{C})^{\mathrm{op}} \simeq \mathrm{VAR}_{\mathscr{J}}$.

$$\begin{tikzcd}
\mathrm{Mnd}_{\mathscr{J}}(\mathscr{C})^{\mathrm{op}} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \mathrm{VAR}_{\mathscr{J}} \arrow[d, hook] \\
\mathrm{Mnd}(\mathscr{C})^{\mathrm{op}} & \mathscr{V}\text{-CAT}/\mathscr{C}
\end{tikzcd}$$

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(More generally, it suffices to suppose $\mathcal{J} \hookrightarrow \mathcal{C}$ is strongly amenable;
see Jason Parker's talk.)

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(2) $$\begin{tikzcd} \mathcal{V} \arrow[d, "\mathcal{V}(I,-)"] \\ \mathrm{Set} \end{tikzcd}$$ topological,

$\mathcal{C} = \mathcal{V}$ or

$\mathcal{C} = \Phi\text{-}\mathrm{Cts}(\mathcal{J}, \mathcal{V})$.

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(+ forthcoming papers)