Differential 2-rigs

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What is a ring?

A ring is a set R equipped with two operations, $+$ and $\times$, such that

- $(R, +)$ is an abelian group with neutral element 0;
- $(R, \times)$ is a monoid with identity 1;
- multiplication is bilinear:

$$\begin{cases} a \times (b + c) = a \times b + a \times c \\ (a + b) \times c = a \times c + b \times c. \end{cases}$$

What is a differential?

On spaces modeled over $\mathbb{R}$ is something like

$$f'x = \lim_{t \to 0} \frac{ft - fx}{t - x}$$

But since we like algebra more than calculus,

Definition

A derivation on a ring R is a linear and Leibniz operator $\partial : R \to R$

- $\partial(a + b) = \partial a + \partial b$;
- $\partial(a \times b) = \partial a \times b + a \times \partial b$.

What is a differential?

As category th'sts / geometers we like differentials because we can represent derivations: one can equip modules (and not just rings) with a notion of derivation and then

Theorem

The functor sending M into $\mathrm{Der}(R, M)$ is representable by a universal module, called object of Kähler differentials, so that

$$\mathrm{Der}(A, M) \cong \mathrm{Mod}_A(\Omega_A, M).$$

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For us, $\mathcal{C}$ has finite coproducts and the canonical maps

$$A \otimes X \cup A \otimes Y \to A \otimes (X \cup Y)$$
$$X \otimes A \cup Y \otimes A \to (X \cup Y) \otimes A$$
$$\varnothing \otimes A \leftarrow \varnothing \to A \otimes \varnothing$$

are all invertible.

What is a differential on a 2-rig?

A derivation D on $\mathcal{C}$ is an endofunctor $D : \mathcal{C} \to \mathcal{C}$, equipped with a left and right tensorial strength, ('applicative structure')

$$\tau^L : DA \otimes B \longrightarrow D(A \otimes B)$$
$$\tau^R : A \otimes DB \longrightarrow D(A \otimes B)$$

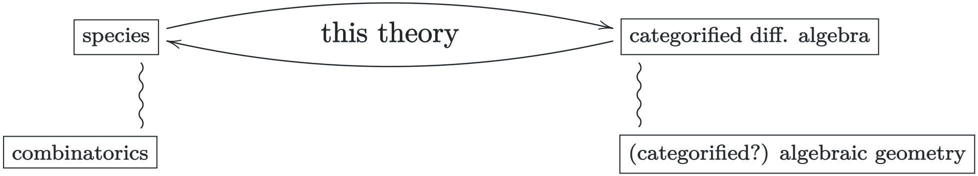
and with the property that the coproduct of the two strengths

$$\begin{bmatrix} \tau^L \\ \tau^R \end{bmatrix} : DA \otimes B \cup A \otimes DB \to D(A \otimes B)$$

(called the 'leibnizator' map) is an isomorphism.

What is this theory about?

- Build a general theory encompassing examples;
- Find new examples of this structure and connections among them;
- Understand what is going on:



Theorem

On a complete category $\mathcal{A}$, strong endofunctors are the coalgebras of a comonad

$$[\mathcal{A},\mathcal{A}] \to [\mathcal{A},\mathcal{A}] : F \mapsto \lambda X. \int_A [A, F(A \otimes X)]$$

As an immediate corollary, we have

Corollary

Every endofunctor of a complete 2-rig is 'best approximated' by a lax derivation

$$\Theta FA \otimes B \xrightarrow{t^L_{AB}} \Theta F(A \otimes B) \xleftarrow{t^R_{AB}} A \otimes \Theta FB$$

Another remarkable result is:

Theorem

Let $\mathcal{C}$ be a 2-rig, and M a internal semigroup with multiplication $m : M \otimes M \to M$; then the map $\partial m : \partial M \otimes M \cup M \otimes \partial M \to \partial M$ splits as a pair of maps

$$\begin{cases} i_R : \partial M \otimes M \to \partial M \\ i_L : M \otimes \partial M \to \partial M \end{cases}$$

Then, i_R (resp., i_L) is a right (resp., left) action of M over ∂M.

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The category of species is the presheaf category over $\mathcal{P}$, 'finite sets and bijections'.

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A florid field of applications: combinatorial differential equations.

- Leroux and Viennot, Labelle, Rajan, Méndez, Nava...

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- take the closure of representables in $[M\Sigma^{\mathrm{op}}, \mathsf{Set}]$ under finite coproducts.

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Theorem

The category of species in a countable set of variables is the free cocomplete differential 2-rig on a single generator.

!!! Parallel with diff. algebra: free objects acquire differential structure.

Given the proper notion of morphism of derivations, one can define a representing object for the derivations on $\mathcal{C}$;

Theorem

There is a universal 2-rig $\mathcal{C}[\epsilon]$ such that

$$\mathsf{Der}(\mathcal{C}) \cong 2\text{-}\mathsf{Rig}(\mathcal{C}, \mathcal{C}[\epsilon]).$$

Moreover, $\mathcal{C}[\epsilon]$ fits into a coinverter diagram

$$\mathcal{C}[Y] \underset{Y^2 \otimes -}{\overset{\varnothing}{\rightrightarrows}} \mathcal{C}[Y] \xrightarrow{q} \mathcal{C}[\epsilon] \qquad (u : \varnothing \Rightarrow Y^2 \otimes -)$$

This parallels differential algebra again; $\mathcal{C}[\epsilon]$ is the 2-rig of Kähler differentials.

Prospects:

- study differential equations;
  - A ‘differential polynomial endofunctor’ (with constant coefficients) is, for example, $X \mapsto X^{(2)} + A \otimes X^{(1)} + X$, where $X^{(n)} := \partial^n X$. Find solutions (=terminal coalgebras) for DPEs: extend theory of polynomial functors.
- study ‘2-rig theory’ more conceptually;
  - find geometric meaning for derivations; extend the theory of quotients we invented to describe Kähler diffs; deformation theory? Galois theory for differential equations?
- find more applications.