Enriched Lenses

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joint work with

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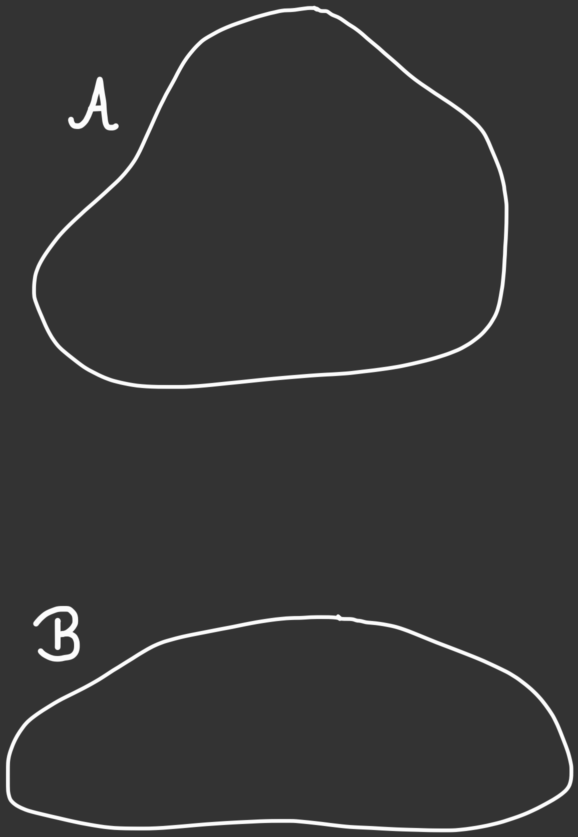
OVERVIEW & MOTIVATION

Lenses model bidirectional transformations:



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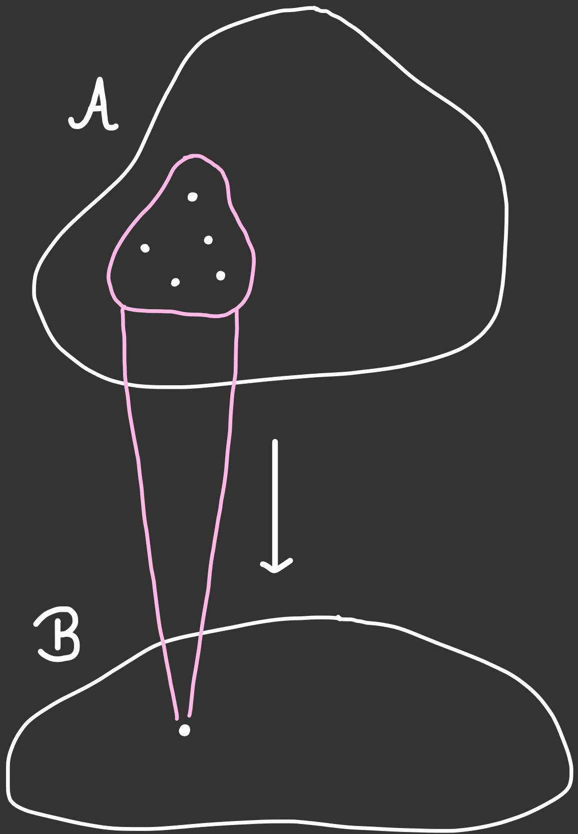

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Lenses model bidirectional transformations:

- object assignment $\rightsquigarrow$ consistency relation
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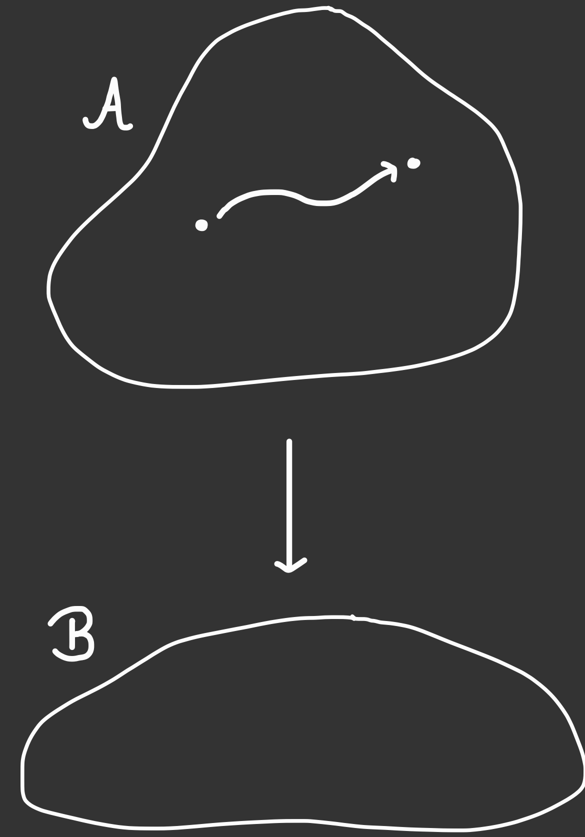

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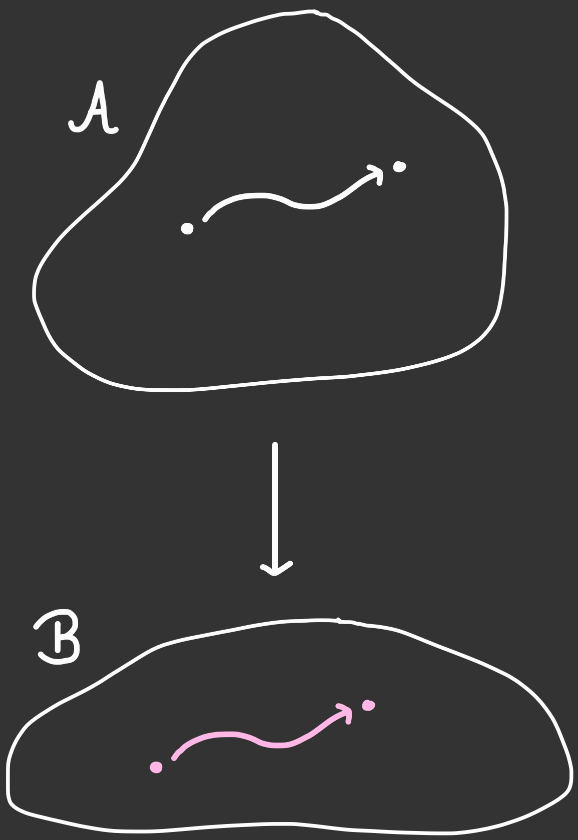

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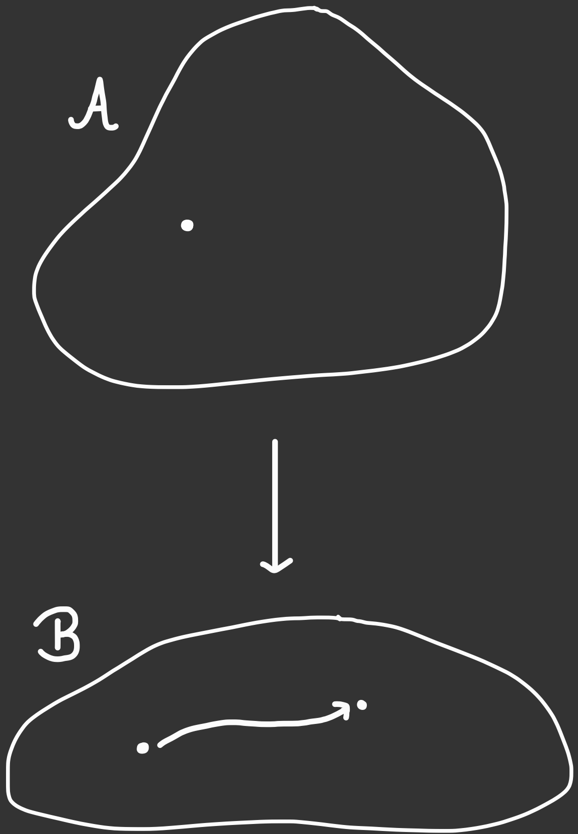

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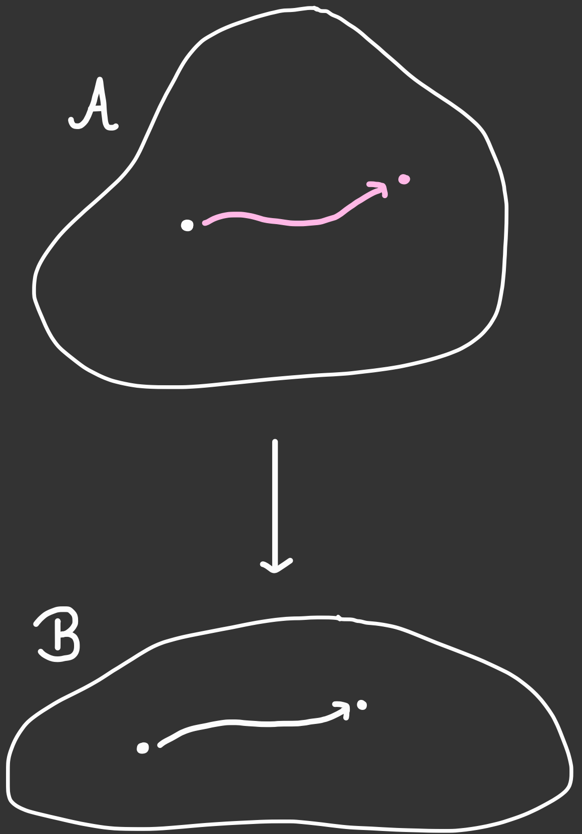

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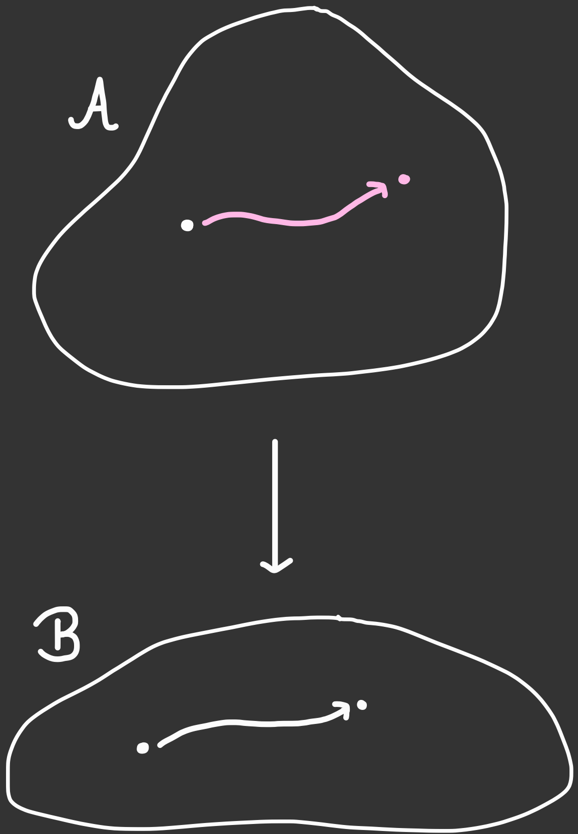

MOTIVATING QUESTION

What is the correct way to restore consistency if the systems are enriched?

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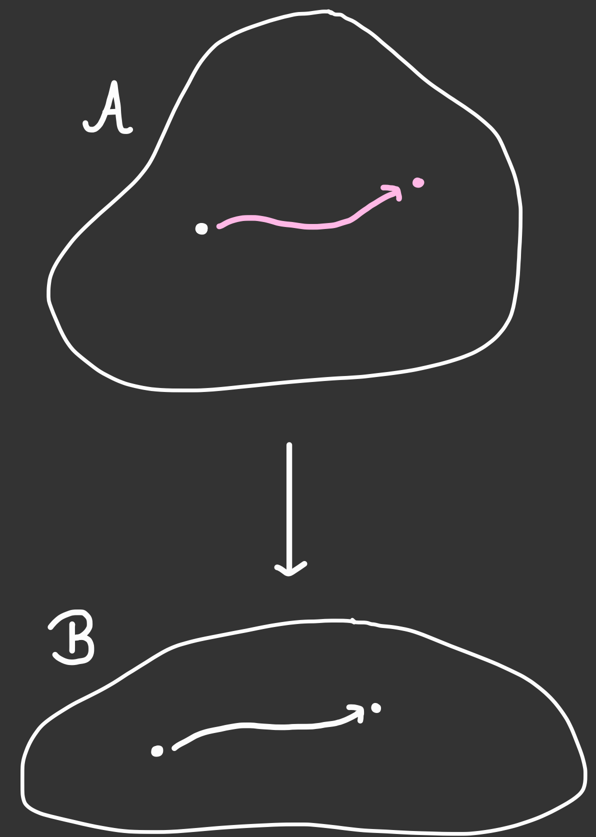

MOTIVATING QUESTION

What is the correct way to restore consistency if the systems are enriched?

1. Introduce enriched cofunctors & lenses.
2. Share some examples, incl. weighted lenses.

02

ENRICHED FUNCTORS VS. COFUNCTORS

Assumption: enrichment in a distributive monoidal category $\mathcal{V}$

enriched functor $F : \mathcal{A} \to \mathcal{B}$

$$\mathrm{obj}(\mathcal{A}) \xrightarrow{F} \mathrm{obj}(\mathcal{B})$$

$$\mathcal{A}(a,a') \xrightarrow{F_{a,a'}} \mathcal{B}(Fa,Fa')$$

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ENRICHED COFUNCTORS AS SPANS

Assumption: enrichment in a distributive monoidal & extensive category $\mathcal{V}$

enriched cofunctor $(F,\varphi) : \mathcal{A} \to \mathcal{B}$ $\simeq$ span of enriched functors:

$$\mathcal{A} \xleftarrow{P} \mathcal{C} \xrightarrow{Q} \mathcal{B}$$

$P : \mathrm{obj}(\mathcal{C}) \to \mathrm{obj}(\mathcal{A})$ invertible

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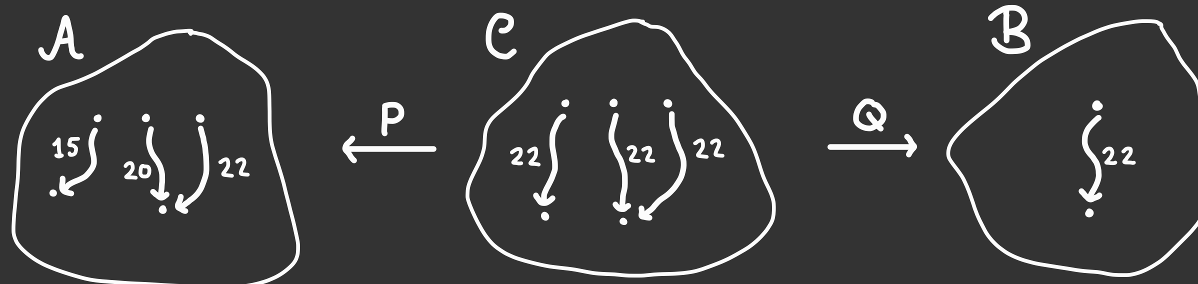

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$$\begin{tikzcd}
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$\mathcal{V} = \omega\mathrm{Set}$ weighted lens

$u \in \mathcal{B}(Fa,b) \quad |\varphi_{a,b}u| = |u|$

05

EXAMPLES OF ENRICHED LENSES

$\mathcal{V}$	Coproduct	lifting map	notes
$(Set, \times, 1)$	$\sqcup$	$\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \bigsqcup_{x \in F^{-1}\{b\}} \mathcal{A}(a,x)$	delta lens Diskin et al. 2011
$(\omega Set, +, 0)$	$\sqcup$	same, s.t. $\lvert \varphi u \rvert = \lvert u \rvert \ \forall u$	weighted lens Perrone 2021
$([0,\infty], +, 0)$	inf	$d(Fa,b) \geqslant \inf_{x \in F^{-1}\{b\}} d(a,x)$	submetry is an example
$(Ab, \otimes, \mathbb{Z})$	$\oplus$ biproduct	$\left\{\mathcal{B}(Fa,b) \xrightarrow{\varphi_{a,b}} \mathcal{A}(a,x)\right\}_{x \in F^{-1}\{b\}}$	additive lens
$Fam(\mathcal{V})$	$\sqcup$	similiar to first example	any monoidal category $\mathcal{V}$

WHY IS THIS NOTION "CORRECT"?

$\mathbb{D}$

double category
with companions

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right-connected completion $\rightsquigarrow$ lenses between monads

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double category with companions	monads, monad morphisms, & monad retromorphisms	right-connected completion $\rightsquigarrow$ lenses between monads
$\mathbb{M}\mathrm{at}(\mathcal{V})$	enriched categories, functors, & cofunctors	enriched lenses
$\mathbb{S}\mathrm{pan}(\mathcal{E})$	internal categories, functors, & cofunctors	internal lenses
$\mathrm{F}\text{-}\mathbb{R}\mathrm{el}$	topological spaces, continuous maps, & open maps	open continuous maps

SUMMARY & FUTURE WORK

enriched lens $\rightsquigarrow$ vertical arrow in $\mathbb{\Gamma}(\mathbb{M}\mathrm{nd}_{\mathrm{ret}}(\mathbb{M}\mathrm{at}(\mathcal{V})))$

$\mathcal{V} = \mathrm{Set}$ $\quad$ $\mathcal{V} = [0, \infty]$

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enriched functor

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$\mathcal{V}$ extensive

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More examples!

- $\mathcal{V} = \mathrm{Poly} \rightsquigarrow$ collectives
- $\mathcal{V} = \mathcal{P}M$, M monoid, $\mathcal{P}$ powerset

More theory!

- Can we better understand enriched split opfibrations using enriched lenses?
- Grothendieck construction for weighted lenses? Enriched?
- 2-cells of enriched cofunctors.