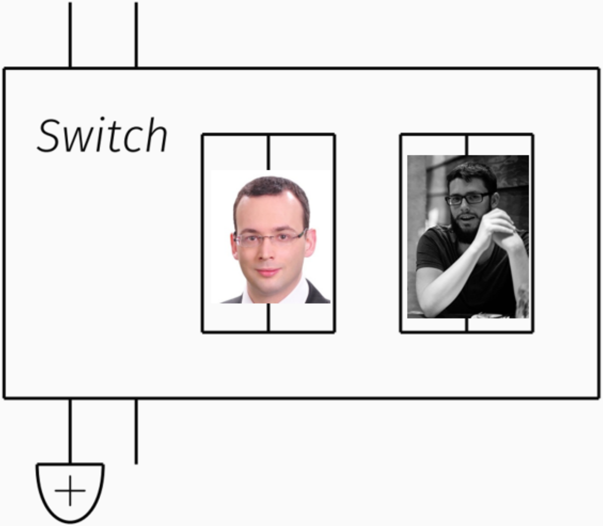
QUANTUM SUPERMAPS ARE CHARACTERIZED BY NATURALITY

Matt Wilson[1,2] Giulio Chiribella[1,2,3,4], Aleks Kissinger[1]

July 22, 2022

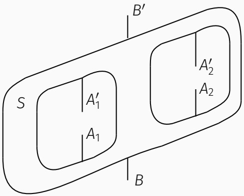
1. Department of Computer Science, University of Oxford, Wolfson Building, Parks Road, Oxford, UK
2. HKU-Oxford Joint Laboratory for Quantum Information and Computation
3. QICI Quantum Information and Computation Initiative, Department of Computer Science
4. Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario, Canada

Black-Box Holes: Supermaps



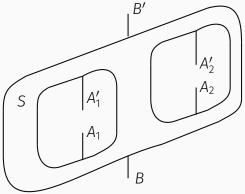
Black-Box Holes: Supermaps

This is a story about holes called **Quantum Supermaps** [1]



Black-Box Holes: Supermaps

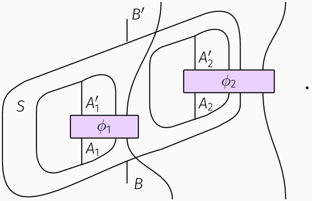
This is a story about holes called **Quantum Supermaps** [1]



$$: \quad \text{Quantum Processes} \to \text{Quantum Processes}$$

Black-Box Holes: Supermaps

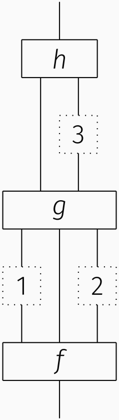
This is a story about holes called **Quantum Supermaps** [1]



Plug in parts of quantum processes $\longrightarrow$ return new quantum processes

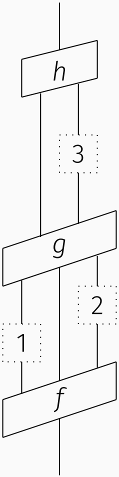
Circuit Theories: With Holes

Circuits with holes: Combs [2]



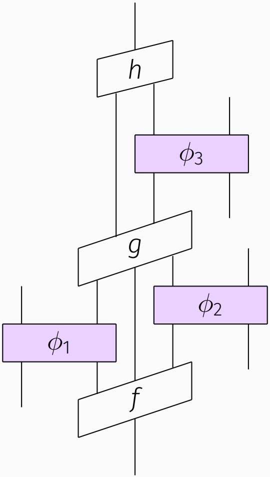
Circuit Theories: With Holes

Circuits with holes: Combs [2]



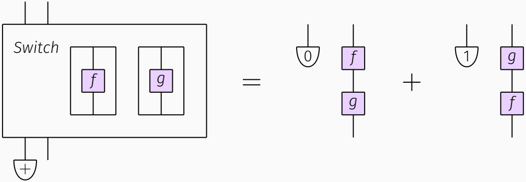
Circuit Theories: With Holes

Circuits with holes: Combs [2]



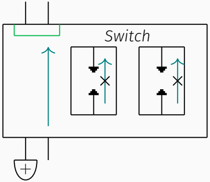
Beyond circuits with definite causal structure

Beyond quantum circuits with holes: The Quantum Switch [3]



Beyond circuits with definite causal structure

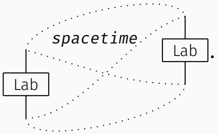
The supermap framework allows to study the impact of causal structure on computation and information processing [4]



Many protocols [5, 6, 7, 3, 8] and real-world experiments [9, 10, 11] have been inspired by the framework

Beyond circuits with definite causal structure

The supermap framework allows to study the global environments which are compatible with localized quantum laboratories

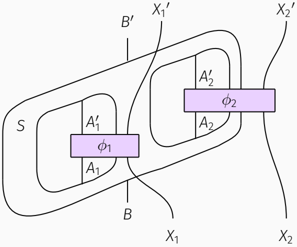


Motivates the study of causal structure via Bell-like inequalities [12]

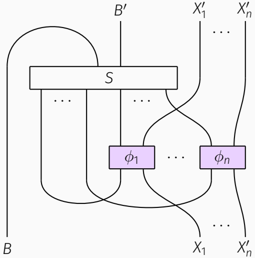
Quantum channels are particular completely positive maps

$$\mathsf{QC} \subseteq \mathsf{CP}$$

Completely positive maps can be used to define supermaps using $\cup, \cap$



$$\approx$$



We have ACT combs, but not ACT Supermaps

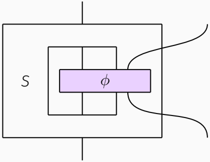
Of the quantum supermaps

- Combs have been successfully generalized to arbitrary symmetric monoidal categories [13, 14, 15, 16]
- To our knowledge, general supermaps have not [**nothing** ...]

This fact made us sad :(

The math is more complicated than the concept :(

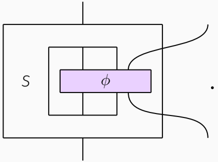
Worse than sad actually, quite perturbed. Lets look back at the picture again,



.

The math is more complicated than the concept :(

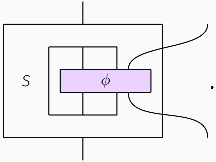
Worse than sad actually, quite perturbed. Lets look back at the picture again,



it **feels** like a purely circuit-theoretic concept

The math is more complicated than the concept :(

Worse than sad actually, quite perturbed. Lets look back at the picture again,



it **feels** like a purely circuit-theoretic concept

- Goal: Find a purely circuit-based characterization of supermaps

Why am I here talking about this?

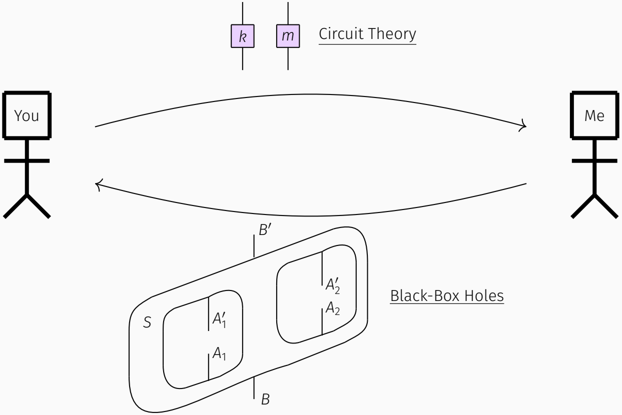
Ok so thats the aesthetic point …But also a structural approach to supermaps is *really needed in physics* …

- We don't know how to define supermaps on arbitrary (monoidal) physical theories
- Even for arbitrary (non-finite) quantum systems there is no agreed-upon definition

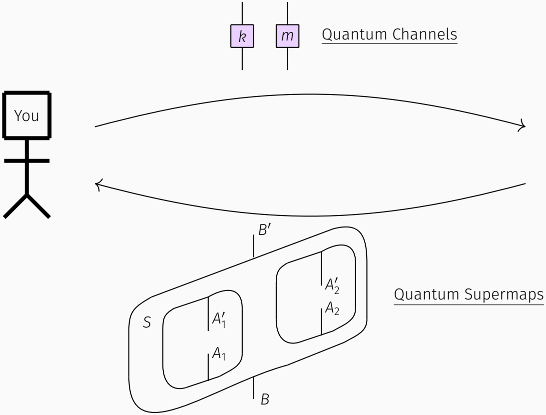
We want a supermap definition which we can use as a hammer on any monoidal category. …Sounds like we need to develop/*apply some category theory*

Circuit-Based Supermap Definition

A first step towards making the categorical supermap hammer

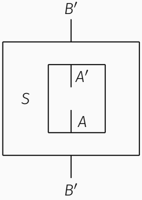


A first step towards making the categorical supermap hammer



Towards a circuit-theoretic definition of supermap

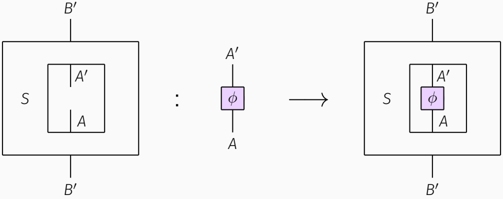
What should a supermap of type $(A \Rightarrow A') \longrightarrow (B \Rightarrow B')$ give us?



$$: \quad (A \Rightarrow A') \longrightarrow (B \Rightarrow B')$$

Towards a circuit-theoretic definition of supermap

What should a supermap of type $(A \Rightarrow A') \longrightarrow (B \Rightarrow B')$ give us?

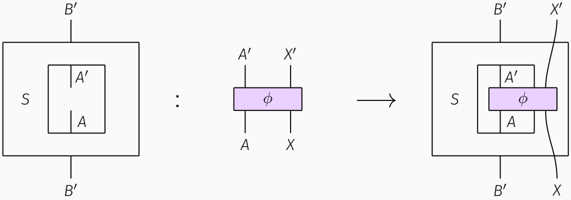


a function

$$S : \mathsf{C}(A, A') \longrightarrow \mathsf{C}(B, B')$$

Towards a circuit-theoretic definition of supermap

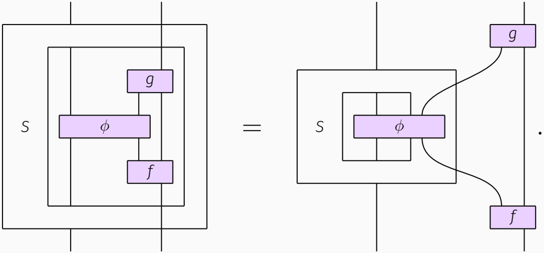
What should a supermap of type $(A \Rightarrow A') \longrightarrow (B \Rightarrow B')$ give us?



a family of functions

$$S_{X,X'} : \mathsf{C}(AX, A'X') \longrightarrow \mathsf{C}(BX, B'X')$$

The principle of locality …commuting with actions on extensions



Can be modelled by relating the $S_{X,X'}$ and $S_{Y,Y'}$ in the following way

$$S_{X,X'}((i \otimes g) \circ (\phi \otimes i) \circ (i \otimes f)) \quad = \quad (i \otimes g) \circ (S_{Y,Y'}(\phi) \otimes i) \circ (i \otimes f)$$

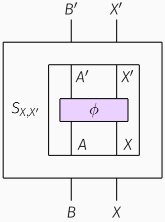
Towards a circuit-theoretic definition of supermap

The family of functions

$$S_{X,X'} : \mathsf{C}(AX, A'X') \to \mathsf{C}(BX, B'X')$$

can be denoted by

$S_{X,X'}(\phi) \quad =:$



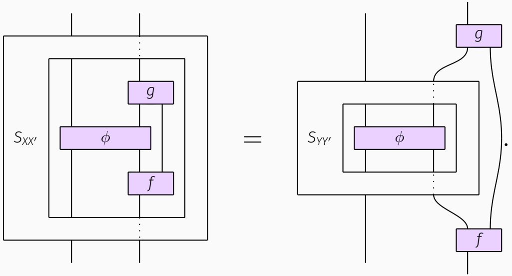
The graphical language is very useful for generalizing to multiple-inputs

The Definition: Locally Applicable Transformations

The locality condition

$$S_{X,X'}((i \otimes g) \circ (\phi \otimes i) \circ (i \otimes f)) \quad = \quad (i \otimes g) \circ (S_{Y,Y'}(\phi) \otimes i) \circ (i \otimes f)$$

is easily parsed in diagrammatic terms



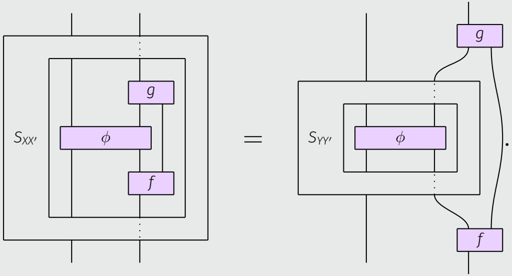
A supermap is like a trace [17, 18] without the yanking

Definition (locally-applicable transformations)

A locally-applicable transformation of type $S : (A \Rightarrow A') \longrightarrow (B \Rightarrow B')$ on a symmetric monoidal category **C** is a family of functions

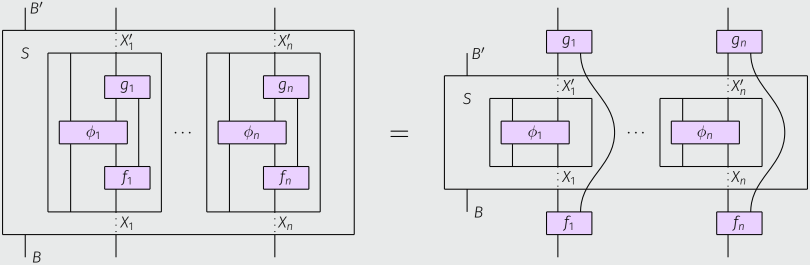
$$S_{XX'} : \mathbf{C}(AX, A'X') \to \mathbf{C}(BX, B'X')$$

such that for every g, f, ϕ then



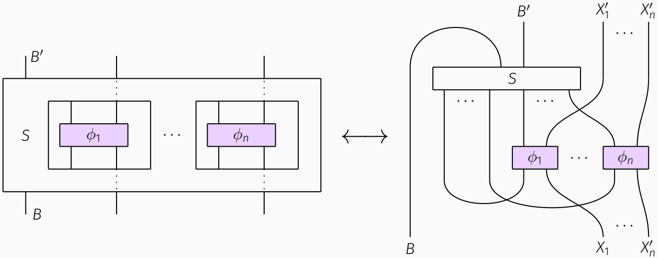
Definition

A *lot* of type $S : (A_1 \Rightarrow A_1') \dots (A_n \Rightarrow A_n') \longrightarrow (B \Rightarrow B')$ is a family of functions $S_{X_1 \dots X_n}^{X_1' \dots X_n'}$ satisfying:



Locality characterizes quantum supermaps!

Any locally-applicable transformation, is implemented by a quantum supermap



Linearity, complete positivity, representation by cups and caps, are all derivable.

Locality characterizes quantum supermaps

Theorem

There is a one-to-one correspondence between locally applicable transformations on quantum channels and deterministic quantum supermaps of type

$$(A_1 \Rightarrow A_1') \ldots (A_n \Rightarrow A_n') \longrightarrow (B \Rightarrow B')$$

In categorical language, there is an equivalence of multicategories

$$\mathsf{lot}[\mathsf{QC}] \cong \mathsf{QS}$$

The Naturality bit

In a locally well-pointed monoidal category, locally applicable transformations are simply natural transformations

$$\mathsf{C}(A-, A' =) \Rightarrow \mathsf{C}(B-, B' =)$$

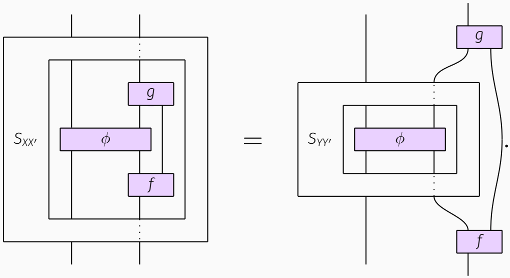
This is true in the multiparty case too.

$$\mathsf{C}(A_1 -_1, A_1{}' =_1) \times \cdots \times \mathsf{C}(A_n -_n, A_n{}' =_n) \Rightarrow \mathsf{C}(B -_1 \cdots -_n, B' =_1 \cdots =_n)$$

So **quantum supermaps** are actually **natural transformations**!!!!

Summary

If you know what a quantum channel is, and you know about natural transformations, or this picture



!!CONGRATS!! You now know what a quantum supermap is

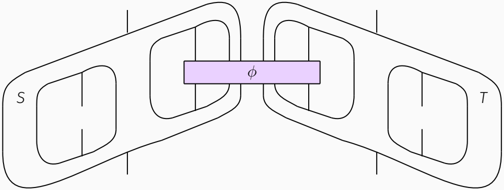
Outlook: Present Directions

Here's some follow-up work on the arxiv [19], a stronger hammer

- A stronger definition of locality for "*polyslots*" satisfies

$$pslot[QC] \cong QS \qquad \mathbf{pslot[uQC]} \cong \mathbf{uQS}$$

- The same strengthening allows us to freely give meaning to



by showing polyslots form a **polycategory**!

Outlook: Future Directions

Here's what we'd like to see next

- Reconstruction of HOQT [20, 21] (the Caus[C] construction) by iterating the definition of locally-applicable transformation
- Use locally-applicable transformations as a model for the most general environments compatible with post-quantum theories
- Characterise the locally-applicable transformations in infinite dimensions in terms of **CPM[$*$Hilb]** [22]
- Development of a category *of* categories of supermaps [23], so that we can use categorical heuristics to identify good supermap definitions
- Does this black box hole perspective appear in other places here at **ACT**?!

Thank-You for Listening!

https://arxiv.org/abs/2205.09844v2

- G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Transforming quantum operations: Quantum supermaps," *EPL (Europhysics Letters)* **83** no. 3, (7, 2008) 30004, *arXiv:0804.0180 [quant-ph]*.
- G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum circuit architecture," *Physical Review Letters* **101** no. 6, (8, 2008) 060401, *arXiv:0712.1325 [quant-ph]*.
- G. Chiribella, "Perfect discrimination of no-signalling channels via quantum superposition of causal structures," *Physical Review A - Atomic, Molecular, and Optical Physics* **86** no. 4, (10, 2012) 040301, *arXiv:1109.5154 [quant-ph]*.
- G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, "Quantum computations without definite causal structure," *Physical Review A - Atomic, Molecular, and Optical Physics* **88** no. 2, (8, 2013) 022318, *arXiv:0912.0195 [quant-ph]*.

D. Ebler, S. Salek, and G. Chiribella, “Enhanced Communication with the Assistance of Indefinite Causal Order,” *Physical Review Letters* **120** no. 12, (3, 2018) 120502.

G. Chiribella, M. Banik, S. S. Bhattacharya, T. Guha, M. Alimuddin, A. Roy, S. Saha, S. Agrawal, and G. Kar, “Indefinite causal order enables perfect quantum communication with zero capacity channels,” *New Journal of Physics* (2, 2021) . *https://iopscience.iop.org/article/10.1088/1367-2630/abe7a0*. arxiv:1810.10457v2.

G. Chiribella, M. Wilson, and H. F. Chau, “Quantum and Classical Data Transmission Through Completely Depolarising Channels in a Superposition of Cyclic Orders,” *arXiv* (5, 2020) , *arXiv:2005.00618 [quant-ph]*.

P. A. Guérin, A. Feix, M. Araújo, and Ĉ. Brukner, “Exponential Communication Complexity Advantage from Quantum Superposition of the Direction of Communication,” *Physical*

Review Letters **117** no. 10, (9, 2016) 100502, arXiv:1605.07372 [quant-ph].

L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. A. Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Ĉ. Brukner, and P. Walther, "Experimental superposition of orders of quantum gates," *Nature Communications* **6** no. 1, (8, 2015) 1–6. www.nature.com/naturecommunications. arxiv:1412.4006.

K. Wei, N. Tischler, S.-R. Zhao, Y.-H. Li, J. M. Arrazola, Y. Liu, W. Zhang, H. Li, L. You, Z. Wang, Y.-A. Chen, B. C. Sanders, Q. Zhang, G. J. Pryde, F. Xu, and J.-W. Pan, "Experimental Quantum Switching for Exponentially Superior Quantum Communication Complexity," *Physical Review Letters* **122** no. 12, (10, 2018) , arXiv:1810.10238v2 [quant-ph].

G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, ?. Brukner, and P. Walther, "Experimental verification of an indefinite causal order," *Science Advances* **3** no. 3, (3, 2017) , arXiv:1608.01683 [quant-ph].

O. Oreshkov, F. Costa, and Ĉ. Brukner, "Quantum correlations with no causal order," *Nature Communications* **3** (2012) , `arXiv:1105.4464 [quant-ph]`.

B. Coecke, T. Fritz, and R. W. Spekkens, "A mathematical theory of resources," `arXiv:1409.5531 [quant-ph]`. `http://arxiv.org/abs/1409.5531http://dx.doi.org/10.1016/j.ic.2016.02.008`.

M. Riley, "Categories of Optics," `arXiv:1809.00738 [quant-ph]`.

M. Román, "Open Diagrams via Coend Calculus," `arXiv:2004.04526 [quant-ph]`.

J. Hefford and C. Comfort, "Coend Optics for Quantum Combs," `arXiv:2205.09027 [quant-ph]`.

A. Joyal and D. Verity, "Traced monoidal categories," *Math. Proc. Camb. Phil. Soc* **119** (2021) 447–451.

N. Pinzani, S. Gogioso, and B. Coecke, "Categorical Semantics for Time Travel," *Proceedings - Symposium on Logic in Computer Science* **2019-June** (1, 2019) , `arXiv:1902.00032 [quant-ph]`.

M. Wilson and G. Chiribella, "Free Polycategories for Unitary Supermaps of Arbitrary Dimension," `arXiv:1412.4006 [quant-ph]`.

A. Bisio and P. Perinotti, "Theoretical framework for higher-order quantum theory," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **475** no. 2225, (5, 2019) 20180706, `arXiv:1806.09554 [quant-ph]`.

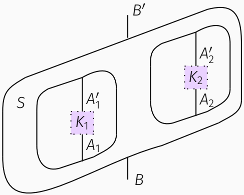
A. Kissinger and S. Uijlen, "A categorical semantics for causal structure," *Logical Methods in Computer Science* **15** no. 3, (2019) , `arXiv:1701.04732 [quant-ph]`.

S. Gogioso and F. Genovese, "Towards quantum field theory in categorical quantum mechanics,". arxiv:1703.09594.

M. Wilson and G. Chiribella, “A Mathematical Framework for Transformations of Physical Processes,” *arXiv:2204.04319 [quant-ph]*.

Theorem

let $K_1, \ldots, K_n, M$ be convex sets of morphisms of QC*, there is a one-to-one correspondence between* CP*-supermaps of type $K_1 \ldots K_n \to M$ and locally-applicable transformations of the same type.*



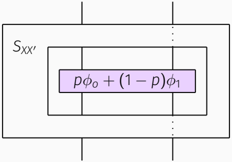
How the proof works

What's the secret sauce?!

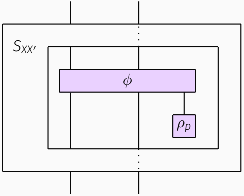
- Convex Linearity of each S is inherited from convex enrichment of the category **QC**
- Consequently S can be extended to the larger circuit theory **CP** of completely positive maps
- Commutation with cups and caps gives internalization

The entire proof works for classical supermaps using the category **fStoch** too!

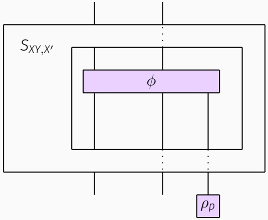
Convex Linearity



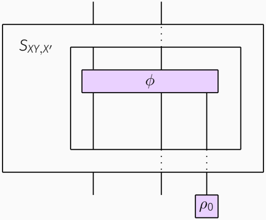
Convex Linearity



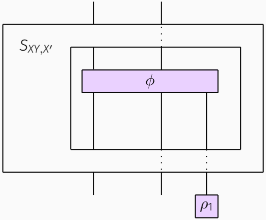
Convex Linearity



$=$ p_o



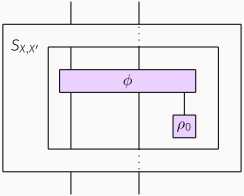
$+$ p_1



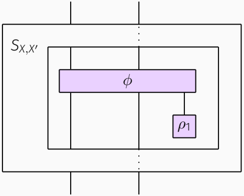
,

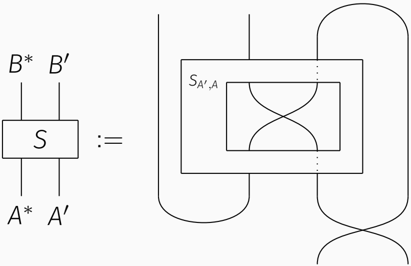
Convex Linearity

$=$ p_0



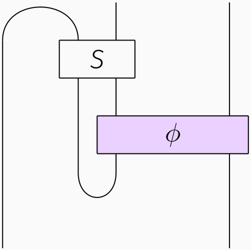
$+$ p_1





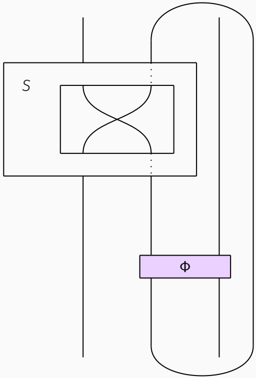
B* B′
S
A* A′
:=
S_{A′,A}

Using cups and caps



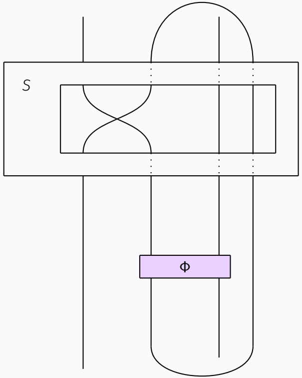
Using cups and caps

=

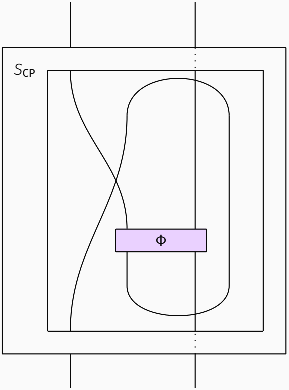


Using cups and caps

$=$

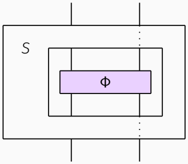


$=$



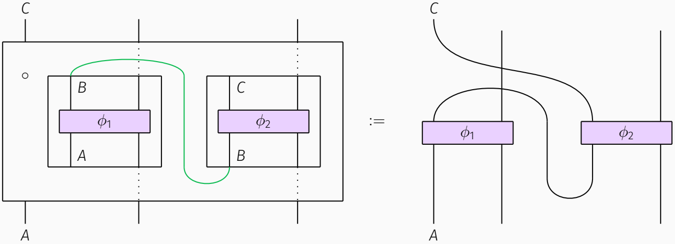
S_{CP}
Φ

Using cups and caps

$=$ 

Enriched Structure for Supermaps

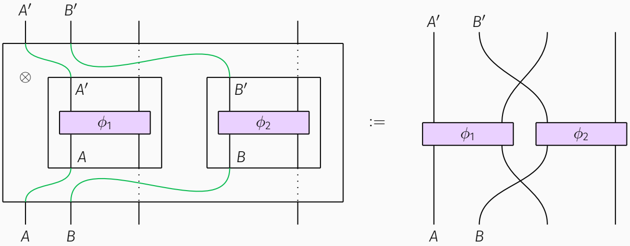
Locally-applicable transformations enrich the category on which they act



Formally have constructed a **lot[C]**-category **C**

Enriched Structure for Supermaps

Locally-applicable transformations enrich the category on which they act



Formally have constructed a **lot[C]**-monoidal category **C**