# Coend Optics, Combs and Quantum Information

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# Outline

Combs "in nature"

Extensional vs intensional combs

CPM construction

(n,m)-combs

# Combs

# Combs in quantum foundations

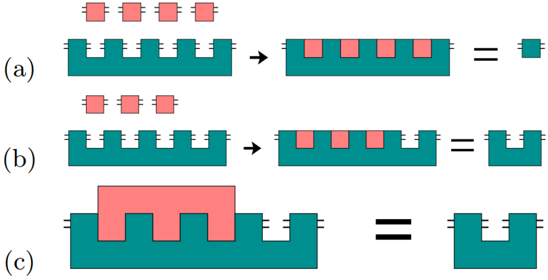



Figure: [Chiribella et al., 2008]

# Combs as maps in a *-autonomous category

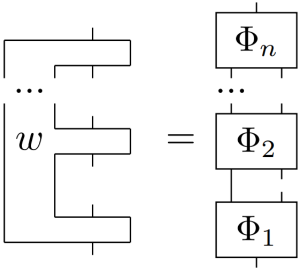



Figure: [Kissinger and Uijlen, 2019]

# Combs as coend optics

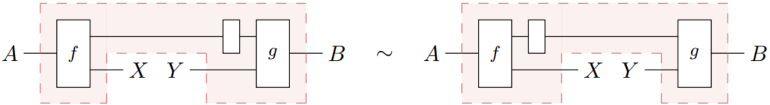



Figure: [Román, 2021]

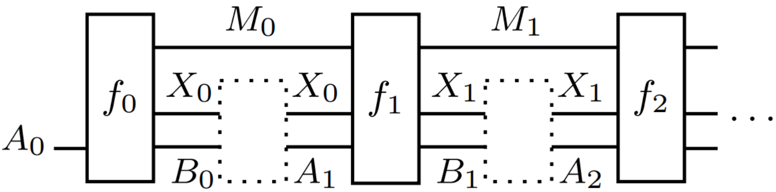


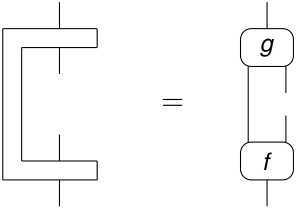

Figure: [Román, 2020]

# Combs in the free cornering construction

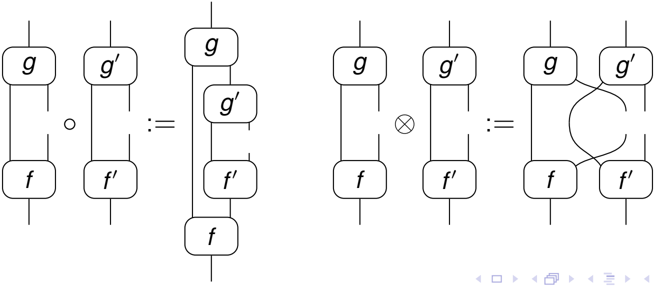

Figure: [Boisseau et al., 2022]

# 2-combs

Consider the case of "2-combs" which factor into a top and bottom part:




We want to give these things the structure of a symmetric monoidal category.



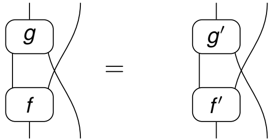
# Extensional combs

We need to quotient by an equivalence relation to get a category.

We want an equivalence relation that captures the extensional behaviour of combs.

There are several candidates, not all are even congruences.

# Extensional combs: i

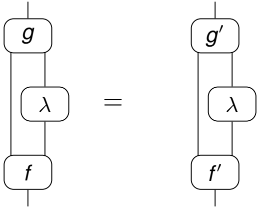
$$(f, g) \sim_\sigma (f', g') \iff$$ 



This is not always a congruence.
Only works in special cases: for example when the base category is compact closed.

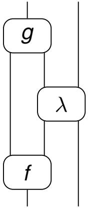
# Extensional combs: ii

$$(f, g) \sim_\tau (f', g') \iff \forall \lambda : B \to B'$$



# Extensional combs: iii

$(f, g) \sim_{\text{comb}} (f', g') \iff \forall\, \lambda$

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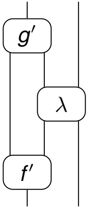
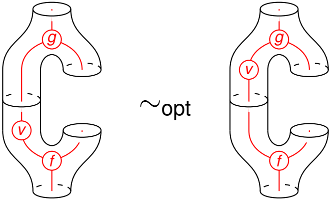
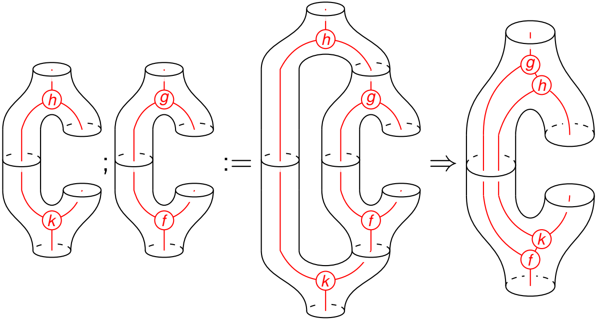


# Intensional combs

Optics are combs inside tubes. Ie. combs modulo:



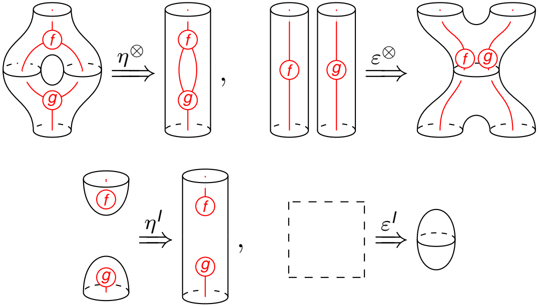

Composition is defined by bubble popping:



# Internal string diagrams for pointed profunctors

Given a monoidal category $\mathcal{C}$, we can put bubbles around the string diagrams by interpreting them in the coslice category $\mathsf{Prof}^* := 1/\mathsf{Prof}$.

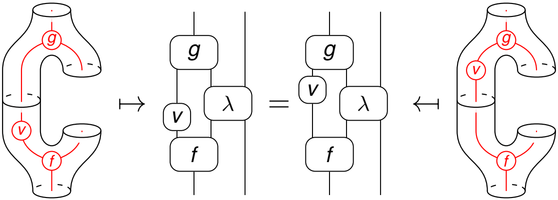
The bubbles come with 2-cells:




Where the pants and copants are pseudo(co)monoids.

# When are these definitions the same?

There is always a functor from $\mathcal{C}$-intensional to extensional combs:

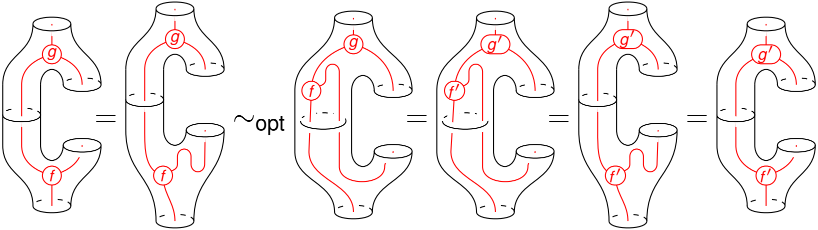



There is not always a functor in the other direction.

We ask when there is one.

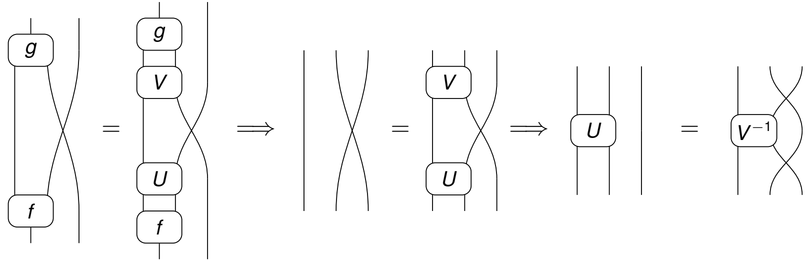
Suppose $(f, g)_E \sim_{\text{comb}} (f', g')_{E'}$...

# Compact closed

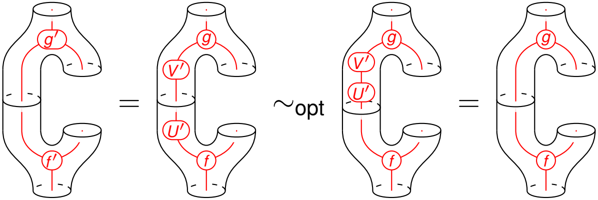


# Unitaries

Because we are in a groupoid there are unitaries $U$, $V$ such that $f' = f; U$ and $g' = V; g$

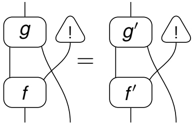



So both $U$, $V$ tensor separate into $U = U' \otimes 1$, $V = V' \otimes 1$. Thus

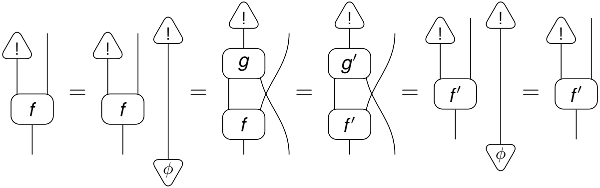


# Inhabited Cartesian

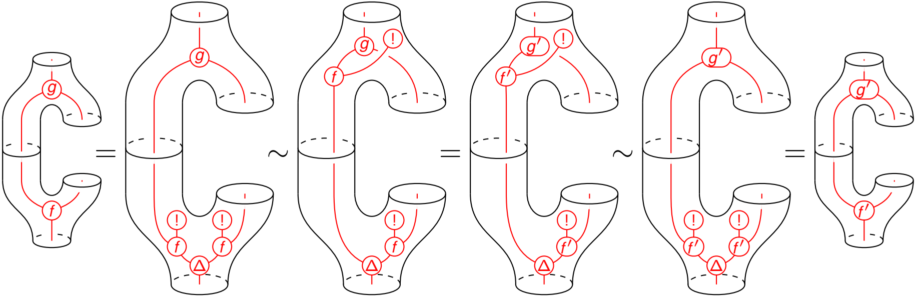
First plug in the swap+discard




For a state $\phi$,



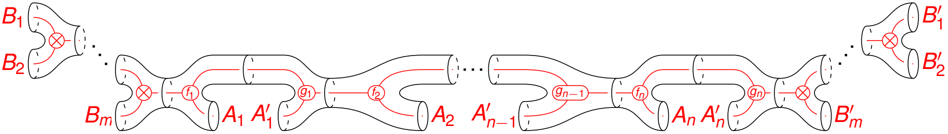

Thus:



# (n,m)-combs

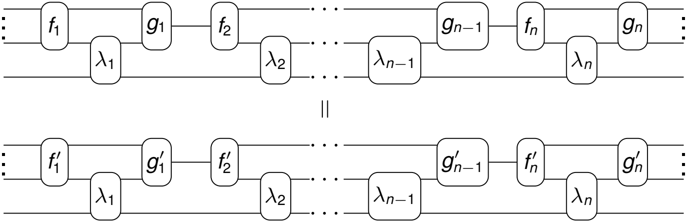
Intensional definition:

$$(\langle f_1, \ldots, f_n | g_1, \ldots, g_n \rangle_{X_1,\ldots,X_n} : [(A_1, A_1'), \ldots, (A_n, A_n')] \to [(B_1, B_1'), \ldots, (B_m, B_m')]) :=$$




Extensional definition:

$\forall \lambda_1, \ldots, \lambda_n$



# Compact closed implies *-polycategory

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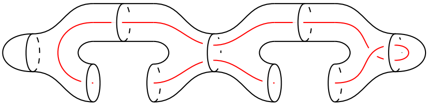
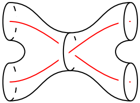
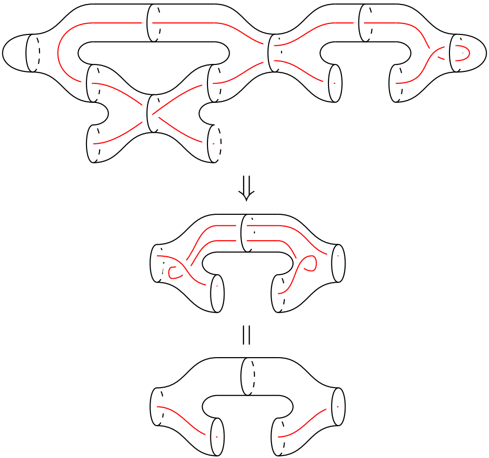
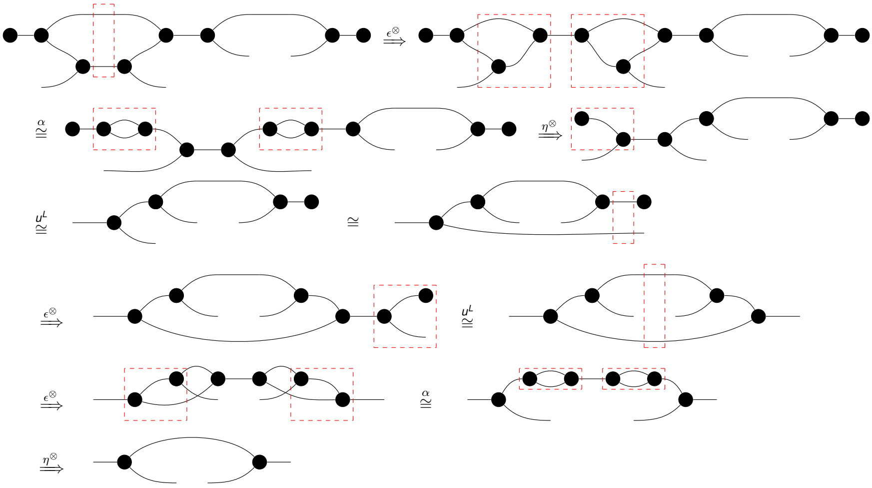


$\varepsilon :=$







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