Making Modalities (lax) Monoidal

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Contributions

Gave modalities of **SLLM** lax monoidal structure

Proved that the proof theory remains intact

Background

Proof Theory

The cut rule:

$$\frac{\Gamma \vdash A \quad \Sigma[A] \vdash B}{\Sigma[\Gamma] \vdash B}\,cut$$

Corresponds to

- *Composition* (categorically speaking)
- *Using lemmas* (logically speaking)

Proof Theory

$$\frac{\Gamma \vdash A \quad \Sigma[A] \vdash B}{\Sigma[\Gamma] \vdash B}\,cut$$

Cut ruins *proof-search*:

i.e. *given* $\Sigma[\Gamma] \vdash B$ *what is its proof?*

How would you guess where A *comes from?*

Cut-elimination: showing that any proof using cut could be done without it!

Lambek Calculus, $\mathbb{L}$

Formulas $A, B ::= p \in \mathrm{Atom} \mid A \bullet B \mid A\backslash B \mid A/B$

Axiom

$$\frac{}{A \vdash A}$$

Left rules

$$\frac{\Gamma[A, B] \vdash C}{\Gamma[A \bullet B] \vdash C}\bullet_L$$

$$\frac{\Gamma \vdash A \quad \Sigma[B] \vdash C}{\Sigma[\Gamma, A\backslash B] \vdash C}\backslash_L$$

$$\frac{\Gamma \vdash A \quad \Sigma[B] \vdash C}{\Sigma[B/A, \Gamma] \vdash C}/_L$$

Right rules

$$\frac{\Gamma \vdash A \quad \Sigma \vdash B}{\Gamma, \Sigma \vdash A \bullet B}\bullet_R$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B}\backslash_R$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A}/_R$$

$\mathbf{L}$ as Grammar

If $\mathrm{Atom} = \{n, s\}$ where n is for *noun* and s is for *sentence*

Consider a dictionary

$$\{dog : n, likes : n\backslash s/n, food : n\}$$

We can prove "*dog likes food*" is a sentence using $\mathbf{L}$:

$$\dfrac{\dfrac{}{n \vdash n} \qquad \dfrac{\dfrac{}{n \vdash n} \quad \dfrac{}{s \vdash s}}{n, n\backslash s \vdash s}}{n, n\backslash s/n, n \vdash s}$$

Cut-elimination for $\mathbf{L}$

A cut-elimination theorem for $\mathbf{L}$ was proven by Lambek in [Lambek '58].

This, in a sense, shows that $\mathbf{L}$ is a category.

It also shows that proof-search for $\mathbf{L}$ is tractable (in fact this is easy).

$\mathbf{L}$ as a Category

$\mathbf{L}$ is a *monoidal bi-closed* category, $\mathscr{C}(\mathbf{L})$ [Sadrzadeh et al '13]

- That is $\mathscr{C}(\mathbf{L})$ is monoidal and we have left and right internal Homs.

$\mathscr{C}(\mathbf{L})$ has $\mathbf{L}$-formulas as objects, and proofs as morphisms.

We distinguish between the logic $\mathbf{L}$ and the category $\mathscr{C}(\mathbf{L})$

Going beyond sentences

$\mathbf{L}$ can parse lots of English, but not more than single sentences/phrases.

We extend this analysis using modalities ($\mathbf{SLLM}$) [McP et al '21]

Inspired by Jäger, Morrill, and Moortgaat to name a few.

SLLM

SLLM

$\mathbf{SLLM} = \mathbf{L} + \{ !, \nabla \}$ [Kanovich et al '20]

That is, $\mathbf{SLLM}$ has the same formulas and rules as $\mathbf{L}$

AND

Has two new kinds of formulas $!A$ and ∇A

AND

new rules that govern $!$ and ∇

SLLM

UCL

Left rules	Right rules	Structural rules
$\dfrac{\Gamma[A, A, \ldots, A] \vdash B}{\Gamma[\,!A] \vdash B}\,!_L$	$\dfrac{A \vdash B}{!A \vdash\, !B}\,!_R$	$\dfrac{\Gamma[B, \nabla A] \vdash C}{\Gamma[\,\nabla A, B] \vdash C}\,\nabla_{E_1}$
$\dfrac{\Gamma[\,\nabla A] \vdash B}{\Gamma[A] \vdash B}\,\nabla_L$	$\dfrac{A \vdash B}{\nabla A \vdash \nabla B}\,\nabla_R$	$\dfrac{\Gamma[\,\nabla A, B] \vdash C}{\Gamma[B, \nabla A] \vdash C}\,\nabla_{E_2}$

SLLM

SLLM lets us parse discourses with references.

For example

$$
\dfrac{
  \dfrac{}{n \vdash n}
  \qquad
  \dfrac{
    \dfrac{}{\nabla n \vdash \nabla n}
    \qquad
    \dfrac{\dfrac{}{n \vdash n} \qquad \dfrac{}{s, s \vdash s, s}}{s, n, n\backslash s \vdash s, s}\ \backslash_L
  }{s,\ \nabla n,\ \nabla n\backslash n, n\backslash s \vdash s, s}\ \backslash_L
}{
  \dfrac{
    \dfrac{
      \dfrac{n, n\backslash s,\ \nabla n,\ \nabla n\backslash n, n\backslash s \vdash s, s}{\nabla n, n\backslash s,\ \nabla n,\ \nabla n\backslash n, n\backslash s \vdash s, s}\ \nabla_L
    }{\nabla n,\ \nabla n, n\backslash s,\ \nabla n\backslash n, n\backslash s \vdash s, s}\ perm
  }{!(\nabla n), n\backslash s,\ \nabla n\backslash n, n\backslash s \vdash s, s}\ contr_2
}\ \backslash_L
$$

Charlie sleeps. She snores.

$\mathscr{C}(\mathbf{SLLM})$

Showed that $\mathscr{C}(\mathbf{SLLM})$ is a monoidal biclosed category equipped with two endofunctors, M and P, for **m**ultiplexing and **p**ermuting respectively.

M and P come with the following natural transformations:

- $\pi_n : M \to \mathrm{id}^{\otimes n}$ for each $n = 1,2,\ldots$ (multiplexing transformation)
- $\sigma : P(-) \otimes \mathrm{id}(=) \cong \mathrm{id}(=) \otimes P(-)$ (permutation)
- $e : P \to \mathrm{id}$ (counit)

Frustrations with $\mathbf{SLLM}$

We didn't require M or P to be compatible with the monoidal product!

But models from [McP et al '21] are lax monoidal.

Monoidal **SLLM**

Recall lax monoidality

A functor between monoidal categories

$$F : (\mathscr{C}, \otimes, I) \to (\mathscr{D}, \boxtimes, J)$$

is lax monoidal when it has the structure (F, m, u) where

- $m : F(-) \boxtimes F(=) \to F(- \otimes =)$ is a natural transformation
- $u : J \to F(I)$ is an arrow in $\mathscr{D}$

Satisfying associativity and unitality equations.

(F is oplax if we reverse all the arrows)

Oplax monoidality in $\mathbf{SLLM}$

Added rules to $\mathbf{SLLM}$ to make M and P oplax monoidal in $\mathscr{C}(\mathbf{SLLM})$

$$\frac{\Gamma[\,!A, !B] \vdash C}{\Gamma[\,!(A \bullet B)] \vdash C}\,!_O \quad \text{and} \quad \frac{\Gamma[\,\nabla A, \nabla B] \vdash C}{\Gamma[\,\nabla(A \bullet B)] \vdash C}\,\nabla_O$$

What about oplax $\mathbf{SLLM}$?

Well, adding the oplax rules ruins the new logic.

This can be seen by the sequent $!\nabla(A \bullet B) \vdash A \bullet (B \bullet B)$.

This sequent has a proof using cut, but cannot be proven without it.

Lax monoidality in $\mathbf{SLLM}$

We added rules to $\mathbf{SLLM}$ to make M and P lax monoidal in $\mathscr{C}(\mathbf{SLLM})$

$$\frac{\Gamma[\,!(A \bullet B)] \vdash C}{\Gamma[\,!A, !B] \vdash C}\,!_M \quad \text{and} \quad \frac{\Gamma[\,\nabla(A \bullet B)] \vdash C}{\Gamma[\,\nabla A, \nabla B] \vdash C}\,\nabla_M$$

This defines the logic $\mathbf{Monoidal\ \ SLLM}$.

Eliminating cut

When adding new rules, one has to eliminate cut from the new system.

Did this in two steps:

1. Show **Monoidal SLLM** is equivalent to **Generalised SLLM**
2. Prove cut-elimination for **Generalised SLLM**

Step 1- equivalence

Generalised SLLM is just **SLLM** where the rules $!_L$, $!_R$ and ∇_R are generalised to allow full structures. We denote the new rules $\tilde{!}_L$, $\tilde{!}_R$ and $\tilde{\nabla}_R$

Usual rules		Generalised rules
$\dfrac{\Gamma[A, A, \ldots, A] \vdash B}{\Gamma[\,!A] \vdash B}\,!_L$	$\rightsquigarrow$	$\dfrac{\Gamma[\Sigma, \Sigma, \ldots, \Sigma] \vdash B}{\Gamma[\,!\Sigma] \vdash B}\,\tilde{!}_L$
$\dfrac{A \vdash B}{!A \vdash\, !B}\,!_R$	$\rightsquigarrow$	$\dfrac{\Gamma \vdash B}{!\Gamma \vdash\, !B}\,\tilde{!}_R$
$\dfrac{A \vdash B}{\nabla A \vdash \nabla B}\,\nabla_R$	$\rightsquigarrow$	$\dfrac{\Gamma \vdash B}{\nabla \Gamma \vdash \nabla B}\,\tilde{\nabla}_R$

Where if

$$\Gamma = A_1, A_2, \ldots, A_n$$

then

$$!\Gamma = \,!A_1, !A_2, \ldots, !A_n$$

(same for ∇)

Step 1- equivalence

Proved equivalence of **Monoidal SLLM** and **Generalised SLLM** by simulating proofs of one in the other and vice versa.

Step 2 - eliminating cut

Proving that cut is eliminable from **Generalised SLLM**.

Used the standard cut-elimination technique of "pushing up" cuts.

This is an induction on the complexity of a proof π.

Show that for every possible combination of rules followed by cut, we can move cut further up in the proof, where it has lower complexity.

Step 2 - eliminating cut

The base case is

$$\dfrac{\overline{A \vdash A} \qquad \overline{A \vdash A}}{A \vdash A}\ cut$$

Which transformed into

$$\overline{A \vdash A}$$

Step 2 - eliminating cut

The principal cuts are ones where we cut along a formula introduced on the left *and* the right. For example:

$$\dfrac{\dfrac{\overline{\Gamma \vdash A}}{\nabla\Gamma \vdash \nabla A}\,\nabla_R \qquad \dfrac{\Sigma, A, \Delta \vdash B}{\Sigma, \nabla A, \Delta \vdash B}\,\nabla_L}{\Sigma, \nabla\Gamma, \Delta \vdash B}\,cut$$

Transformed into

$\rightsquigarrow$

$$\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\overline{\Gamma \vdash A} \qquad \overline{\Sigma, A, \Delta \vdash B}}{\Sigma, \Gamma, \Delta \vdash B}\,cut}{\Sigma, A_1, A_2, \ldots, A_m, \Delta \vdash B}}{\Sigma, \nabla A_1, A_2, \ldots, A_m, \Delta \vdash B}\,\nabla_L}\,\nabla_L}{\Sigma, \nabla A_1, \nabla A_2, \ldots, A_m, \Delta \vdash B}\,\nabla_L}{\dfrac{\Sigma, \nabla A_1, \nabla A_2, \ldots, \nabla A_m, \Delta \vdash B}{\Sigma, \nabla\Gamma, \Delta \vdash B}}\,\nabla_L$$

Step 2 - eliminating cut

The non-principal cuts are ones where the cut formula is introduced by only the left or right sequent. For example

$$\dfrac{\dfrac{\overline{A, \Gamma \vdash B}}{\Gamma \vdash A \backslash B}\ \backslash_R \qquad \dfrac{\overline{\Sigma, C, D, \Delta \vdash E}}{\Sigma, C \cdot D, \Delta \vdash E}\ \cdot_L}{\Sigma, C \cdot D, \Delta \vdash E}\ cut$$

Transformed into

$\rightsquigarrow$

$$\dfrac{\dfrac{\dfrac{\overline{A, \Gamma \vdash B}}{\Gamma \vdash A \backslash B}\ \backslash_R \qquad \overline{\Sigma, C, D, \Delta[A \backslash B] \vdash E}}{\Sigma, C, D, \Delta[\Gamma] \vdash E}\ cut}{\Sigma, C \cdot D, \Delta[\Gamma] \vdash E}\ \cdot_L$$

Summary & Outlook

We have shown that

1. Adding lax monoidal structure to $\mathbf{SLLM}$ preserves its proof theory.

2. Adding oplax monoidal structure completely ruins its proof theory.

This takes us one step closer to

3. Finding a complete model of $\mathbf{SLLM}$

4. Defining a graphical language.

Thank you for listening

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