Fiblang

Fibrational approach to categorical linguistics

Genovese, Loregian, Puca

ACT2022

Introduction

Introduction to Fiblang

An unbiased formalization for linguistics

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An unbiased formalization for linguistics

Definition (Speaker)

A *speaker* is a functor $p : \mathcal{D}^p \to \mathcal{L}$.

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An unbiased formalization for linguistics

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We know nothing about this $\dashrightarrow \mathcal{D}^p$

$$\mathcal{D}^p \xrightarrow{\;p\;} \mathcal{L}$$

$\mathcal{L} \dashleftarrow$ This can be described

Introduction to Fiblang

Fibrations

Introduction to Fiblang

Fibrations

Definition (Fibration)

A functor $p^\sharp : \mathcal{E} \to \mathcal{C}$ is a *fibration* if, for every object E in $\mathcal{E}$ and every morphism $f : C \to p^\sharp E$ with C in $\mathcal{C}$, there exists a unique morphism $h : E' \to E$ such that $p^\sharp h = f$.

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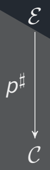

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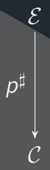

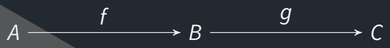

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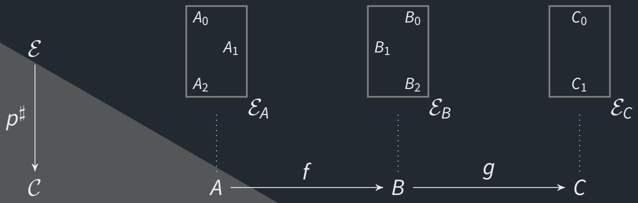

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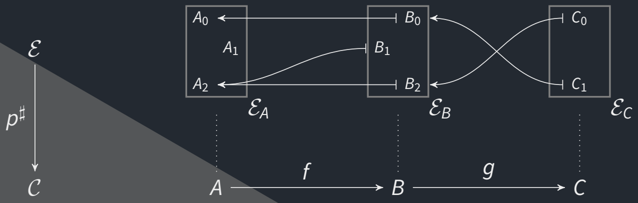

Introduction to Fiblang

Main theorems

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Main theorems

Theorem

Any functor $p : \mathcal{D}^p \to \mathcal{L}$ can be written as a composition of functors $\mathcal{D}^p \xrightarrow{s} \mathcal{E}^p \xrightarrow{p^\sharp} \mathcal{L}$, such that $p^\sharp$ is a fibration.

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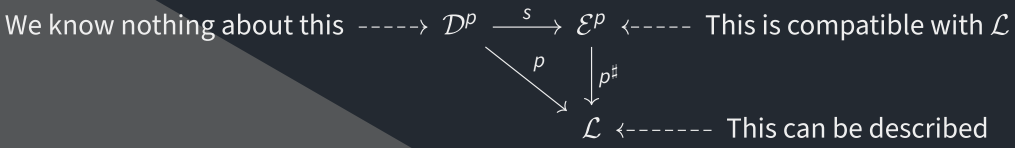

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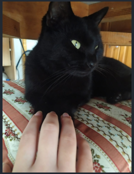
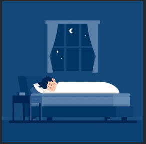
Toy example

Introduction to Fiblang

Toy example

$$\mathcal{D}^p \xrightarrow{s} \mathcal{E}^p, \quad p: \mathcal{D}^p \to \mathcal{L}, \quad p^\sharp: \mathcal{E}^p \to \mathcal{L}$$





'the cat sleeps'

Language Acquisition

Language Acquisition

What makes us human

Language Acquisition

What makes us human

The Final Wave:
***Homo sapiens* biogeography and the evolution of language**

Telmo Pievani
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"Language was central to human expansion across the globe. It was our secret weapon, and as soon we got language we became a really dangerous species"

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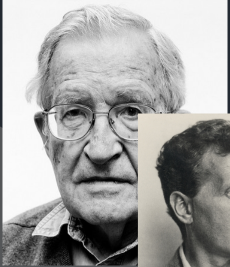
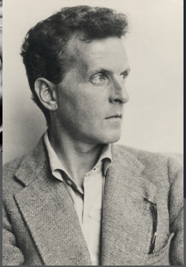
How is that?

Language Acquisition

The problem of language acquisition





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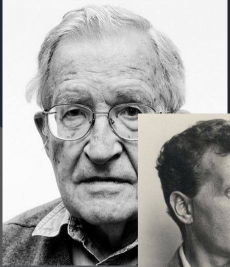
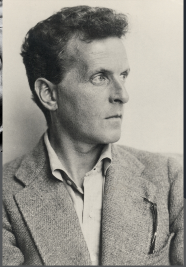

- Mentalism vs. Behavourism
- Innateness Hypothesis vs. Language Games

Language Acquisition

Let's play language games!

BUILDER: Slab!

Language Acquisition

Let's play language games!

BUILDER: Slab!

ASSISTANT: (*handles a beam*)

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Let's play language games!

BUILDER: Slab!

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BUILDER: No! Slab! (*pointing at a slabVocabulary acquisition by example*)

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Let's play language games!

BUILDER: Slab!

ASSISTANT: (*handles a beam*)

BUILDER: No! Slab! (*pointing at a slabVocabulary acquisition by example*)

ASSISTANT: (*understands the correction, handles a slab*)

BUILDER: Right.

Vocabulary acquisition by example

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Formally

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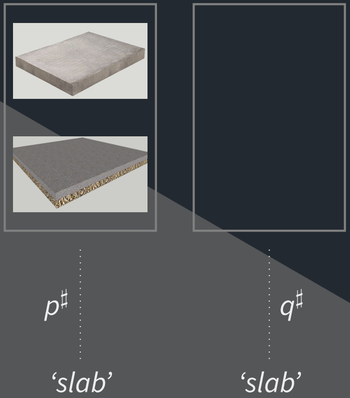

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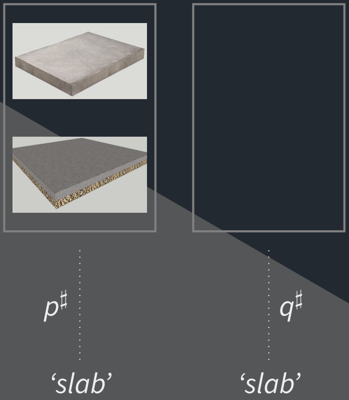

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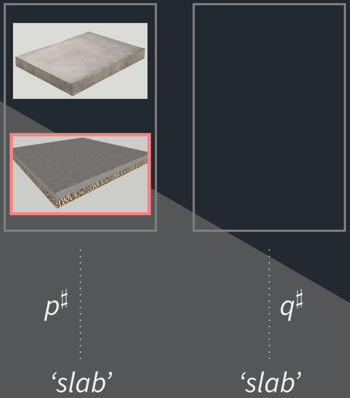

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$$\mathrm{obj}(\mathcal{F}^q) := \mathrm{obj}(\mathcal{E}^q) \sqcup S$$

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Vocabulary acquisition by example

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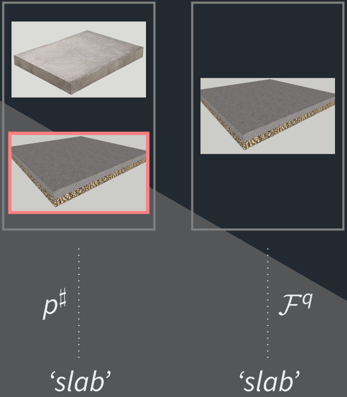

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Define $T : \mathcal{F}^q \to \mathcal{L}$ such that it agrees with q.

Vocabulary acquisition by example

Define $T : \mathcal{F}^q \to \mathcal{L}$ such that it agrees with q.

Are we done?

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Are we done? No!

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We need new knowledge to be **compatible** with the previous.

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Define $T : \mathcal{F}^q \to \mathcal{L}$ such that it agrees with q.

Are we done? No!

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By Factorisation Theorem:

$$\begin{array}{ccc} \mathcal{F}^q & \xrightarrow{\;s\;} & \mathcal{E}^{\tilde{q}} \\ & \searrow^{T} & \downarrow{\scriptstyle \tilde{q}} \\ & & \mathcal{L} \end{array}$$

Vocabulary acquisition by paraphrasis

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A simple example

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ALICE: I adopted a cat!

Vocabulary acquisition by paraphrasis

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ALICE: I adopted a cat!

BOB: A what?

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ALICE: Well, almost; black cats are cursed.

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ALICE: Well, almost; black cats are cursed.

BOB: Ah, now I see.

Vocabulary acquisition by paraphrasis

Explanations

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Theorem

There is an equivalence of categories $\nabla - : \mathit{DFib}/\mathcal{L} \cong [\mathcal{L}, \mathit{Set}] : \int -$.

Vocabulary acquisition by paraphrasis

Explanations

Theorem

There is an equivalence of categories $\nabla - : DFib/\mathcal{L} \cong [\mathcal{L}, Set] : \int -$.

Definition (Explanation)

Consider a speaker $p^\sharp : \mathcal{E}^p \to \mathcal{L}$ and an object L of $\mathcal{L}$. An *explanation for L according p* is a finite diagram $D_L : \mathcal{A} \to \mathcal{L}$ such that the limit $\hat{L}$ of the diagram

$$\mathcal{A} \xrightarrow{D_L} \mathcal{L} \xrightarrow{\nabla p^\sharp} \mathsf{Set}$$

is a subset of the fibre $\mathcal{E}^p_L$.

Vocabulary acquisition by paraphrasis

Explanations, visually

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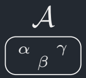
Explanations, visually



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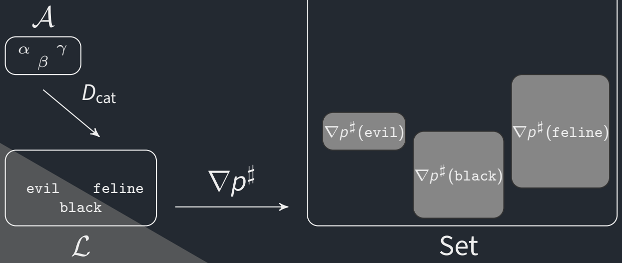
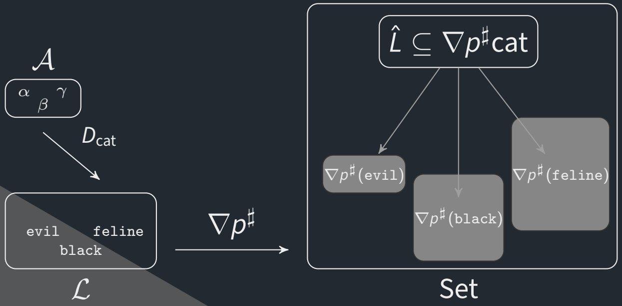


Vocabulary acquisition by paraphrasis

Explanations, visually



Vocabulary acquisition by paraphrasis

A first endeavour

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- $\mathcal{E}^p_L \neq \emptyset$ (p *knows* the meaning of L);
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$$\mathcal{A} \xrightarrow{D_L} \mathcal{L} \xrightarrow{\nabla q^\sharp} \mathrm{Set}$$

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Problem: $\hat{L}$ is a cone but including its legs in $\mathcal{F}^q$, a functor $\mathcal{F}^q \to \mathcal{L}$ may not be definable.

Vocabulary acquisition by paraphrasis

Solution: extend the language!

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Define a quiver Q as follows:

Vocabulary acquisition by paraphrasis

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Vocabulary acquisition by paraphrasis

Solution: extend the language!

Define a quiver Q as follows:

- Vertexes of Q are objects of $\mathcal{L}$;
- There is an edge $L \to L'$ in Q iff there is a morphism $\phi : \hat{L} \to \nabla q^\sharp L'$ in the limiting cone of $\mathcal{A} \xrightarrow{D_L} \mathcal{L} \xrightarrow{\nabla q^\sharp}$ Set.

Vocabulary acquisition by paraphrasis

Solution: extend the language!

Define a quiver Q as follows:

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Now, we want to 'glue' the quiver Q to the category $\mathcal{L}$.

Vocabulary acquisition by paraphrasis

Glueing Q to $\mathcal{L}$

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Consider the adjunction $F : \text{Quiv} \rightleftarrows \text{Cat} : U$, and define the category $\mathcal{L} \wr FQ$ as:

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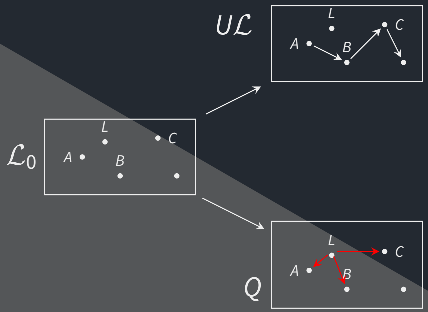

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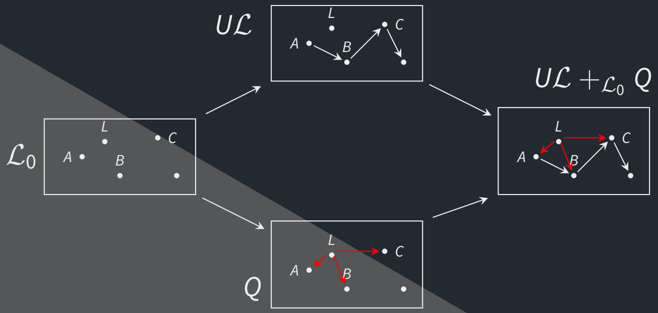

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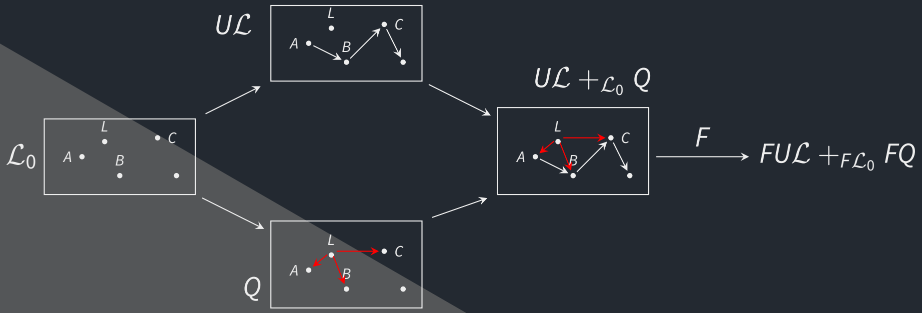

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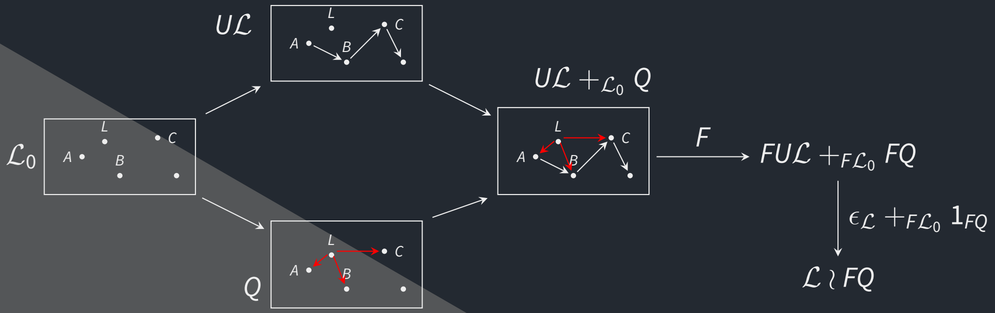
Glueing Q to $\mathcal{L}$

Consider the adjunction $F : \text{Quiv} \rightleftarrows \text{Cat} : U$, and define the category $\mathcal{L} \wr FQ$ as:



Vocabulary acquisition by paraphrasis

Finishing touches

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Now we can define a functor $T : \mathcal{L} \wr FQ \to \text{Set}$ as follows:

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- On objects:
  - We map L to $\hat{L}$;

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- On morphisms:

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The new fibration modelling q after having learned L is $\int T$.