Exchangeability and the Radon Monad

Probability Measures, Quantum States and Multisets

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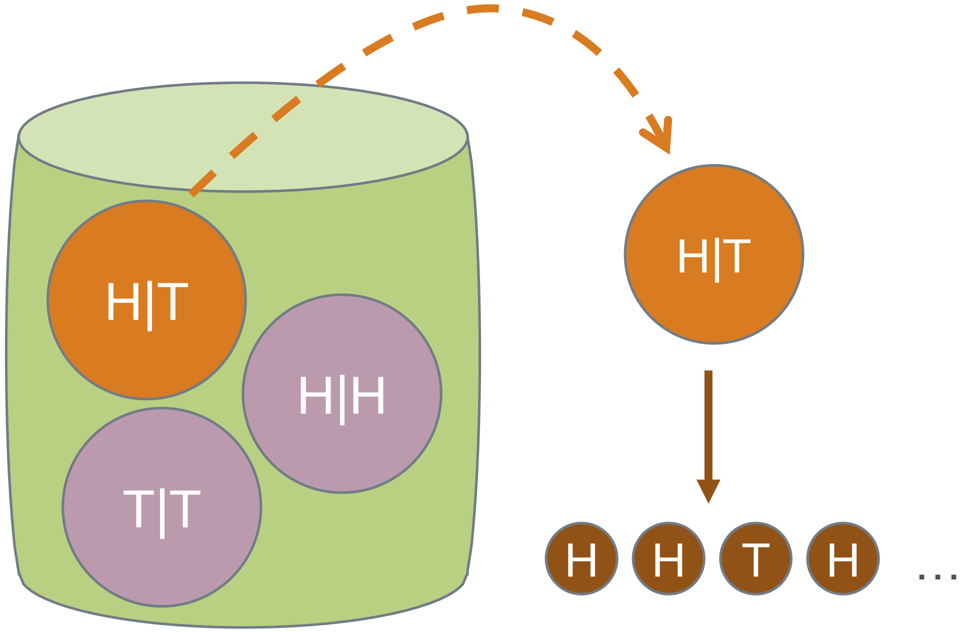
De Finetti's Theorem

› Imagine a bag of coins…



$$\mathbb{P}(H,T) = \mathbb{P}(T,H) = \frac{1}{3} \times \frac{1}{4}$$

$$\mathbb{P}(H,H,H) = \frac{1}{3} + \frac{1}{3} \times \frac{1}{8}$$

$$\mathbb{P}(T,H,H) = \mathbb{P}(H,T,H) = \mathbb{P}(H,H,T) = \frac{1}{3} \times \frac{1}{8}$$

› Flips are not quite independent…

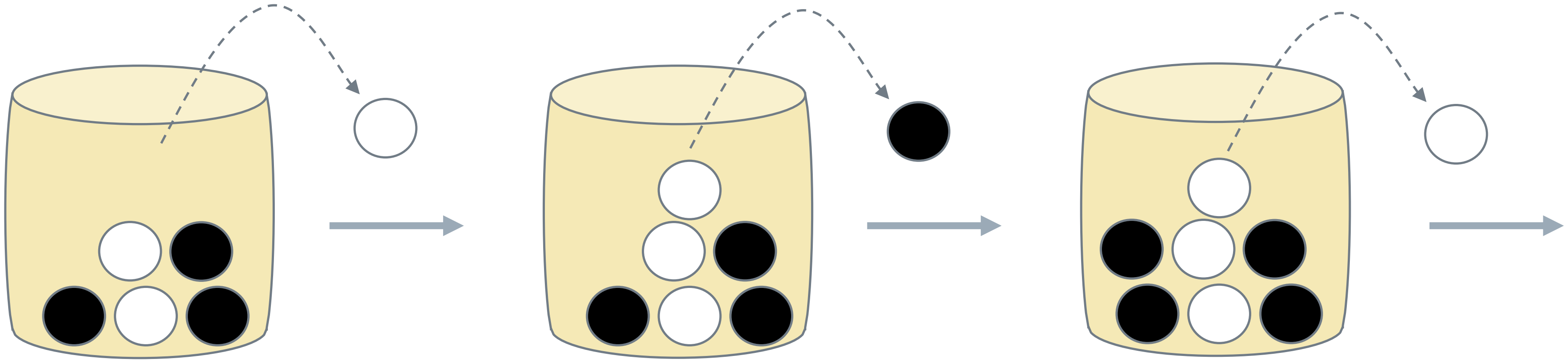
Classical De Finetti Theorem:

every exchangeable sequence is drawing a random measure from a bag and repeating it forever!

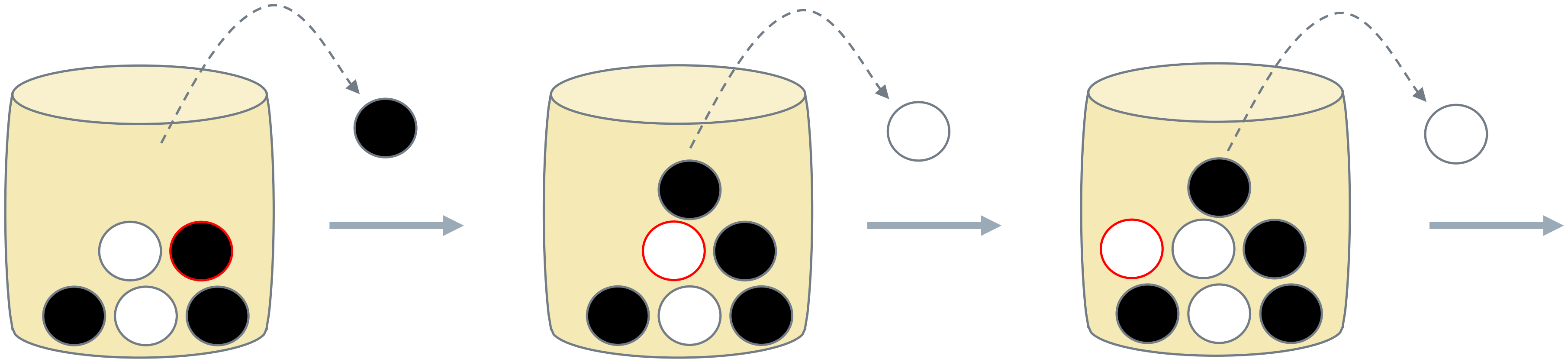
De Finetti, 1937; Hewitt, Savage, 1955

Polya's Urn

$n = 2, k = 3$



Start with n white balls, and k black balls. Draw a ball, record the result and replace with two; continue…
Exchangeable!



Polya's Urn

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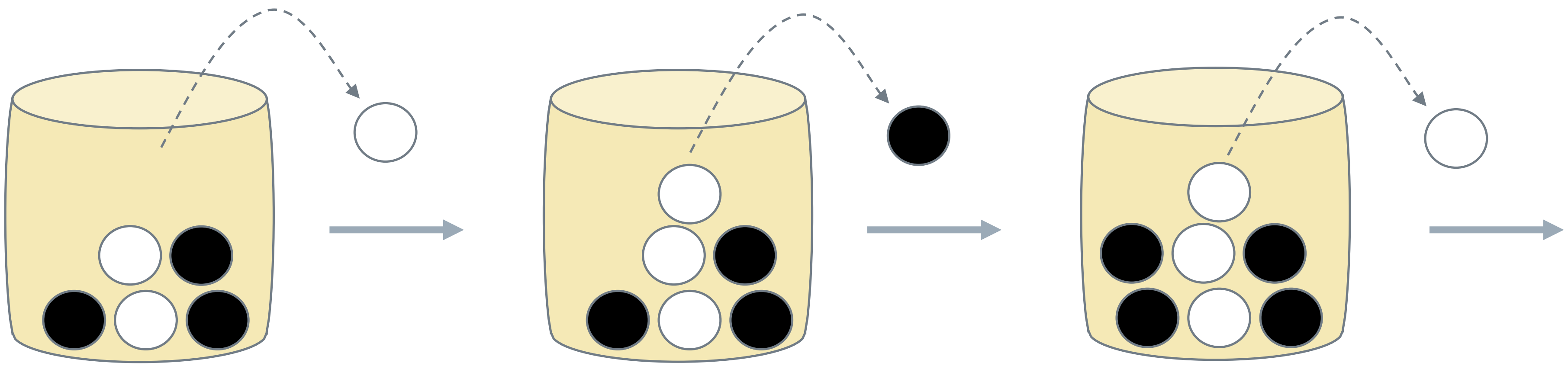
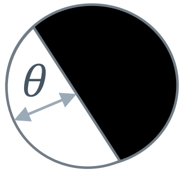


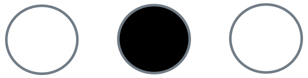
Start with n white balls, and k black balls. Draw a ball, record the result and replace with two; continue…

Exchangeable! What is the De Finetti measure?

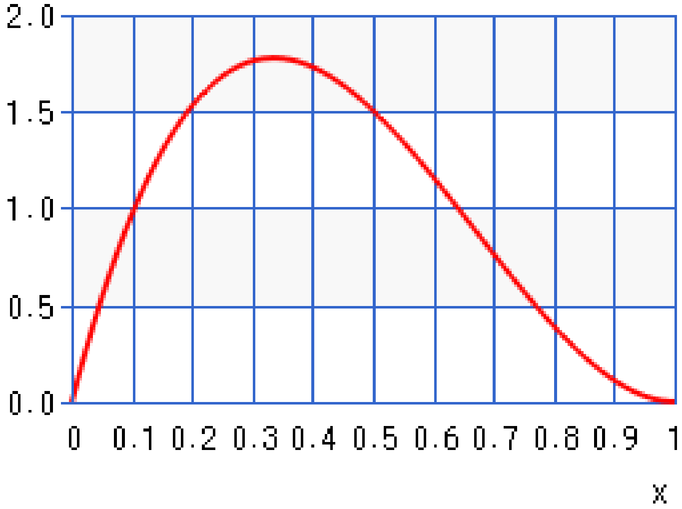
Pick a black-white coin with bias $\theta \sim \mathrm{Beta}(n, k)$



Flip repeatedly: $F_i \sim \mathrm{Bernoulli}(\theta)$



…



Radon Measures

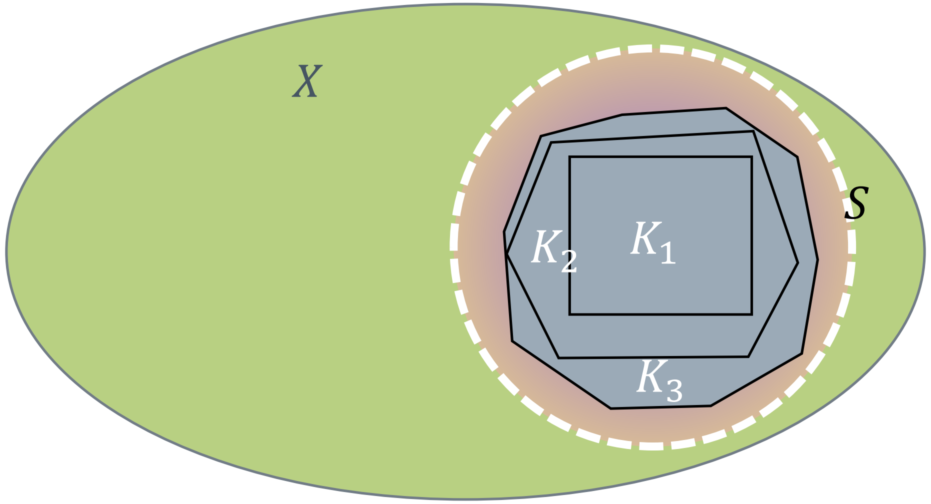
No "escaping to infinity"

Points are "nicely" separated

Let X be a compact, Hausdorff topological space + Borel σ-algebra.

A probability measure μ on X is *Radon* if the measure of a set is given by measuring compact subsets:

$$\mu(S) = \sup_{\substack{K \subset S \\ K \text{ measurable} \\ +\text{compact}}} \mu(K)$$



Compact Haus. Spaces + Cont. Maps

In $\mathbf{CH}$, measures that are determined by subobjects.

The Radon Monad

Compact Haus. Spaces + Cont. Maps

$$\mathcal{R}\colon \mathbf{CH} \to \mathbf{CH}$$

On Spaces: space of measures

$$X \mapsto \mathcal{R}X := \{\mu \mid \text{Radon measures on } X\}$$

Topologised by all $\{\mu \mid \int_X f \ \mathrm{d}\mu \subset \Omega\}$

for $f : X \to \mathbb{C}$ continuous and $\Omega \subset \mathbb{C}$ open

Measures are close if they integrate lots of functions to similar values

On Continuous Maps: pushforward measures

$$(f : X \to Y) \mapsto (\lambda\mu. f_*\mu : \mathcal{R}X \to \mathcal{R}Y)$$

Unit of Monad: delta distributions

$$X \to RX$$
$$x \mapsto \delta_x$$

Multiplication of Monad: averaging

$$\mathcal{R}^2X \to \mathcal{R}X$$
$$\Phi \mapsto \left(\lambda S. \int_{\mu\in\mathcal{R}X} \mu(S)\mathrm{d}\Phi\right)$$

“Is S likely?” ↔“Am I likely to draw a measure for which S is likely?”

Radon Kleisli Maps and Markov Categories

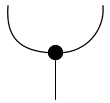
The category $\mathrm{Kl}(\mathcal{R})$:

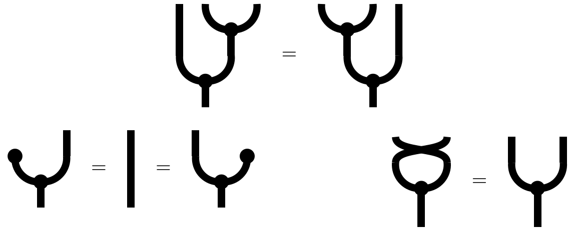
Objects: $X \in \mathbf{CH}$

Morphisms: $X \rightsquigarrow Y \equiv X \to \mathcal{R}Y$

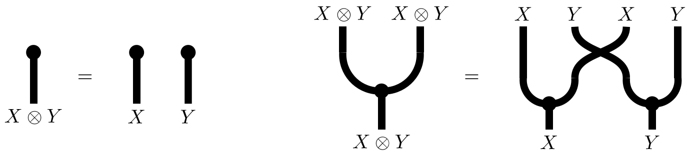
Markov Categories: semicartesian symmetric monoidal $(\mathcal{C},\ \otimes,\ I)$

+ Commutative Comonoids

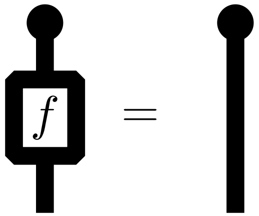
$\mathsf{copy}_X =$  $\mathsf{del}_X =$



as well as compatibility with the monoidal structure,



Naturality of del_X



Naturality of copy_X?

Not with probability!

Fritz, 2019

Exchangeability Categorically

Let $X \in \mathbf{CH}$: How to describe an exchangeable measure?

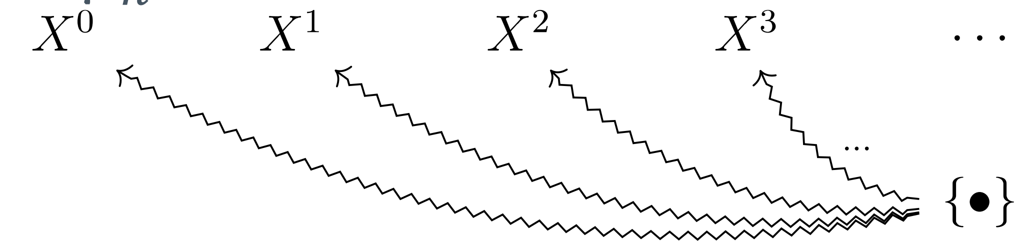
- A measure μ on $X^{\mathbb{N}}$
- such that, for any projection $\pi_n : X^{\mathbb{N}} \to X^n$ and any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$,

$$(\pi_n)_* \mu = (X^\sigma \pi_n)_* \mu$$

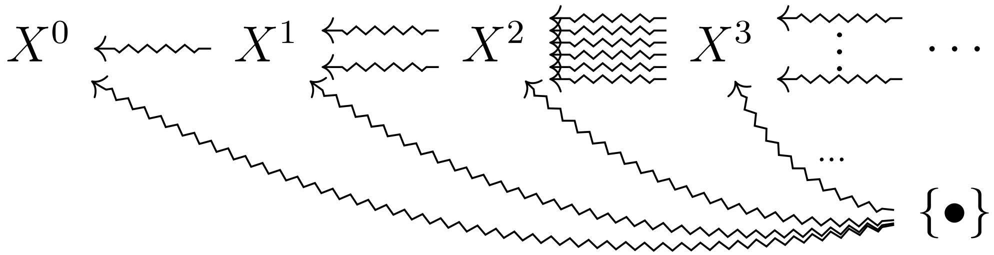
Exchangeability Categorically

Let $X \in \mathbf{CH}$: How to describe an exchangeable measure?

- A set of measures μ_n on X^n for each $n \in \mathbb{N}$



- which are compatible: for $\pi_{mn}: X^m \to X^n, (\pi_{mn})_* \mu_m = \mu_n$



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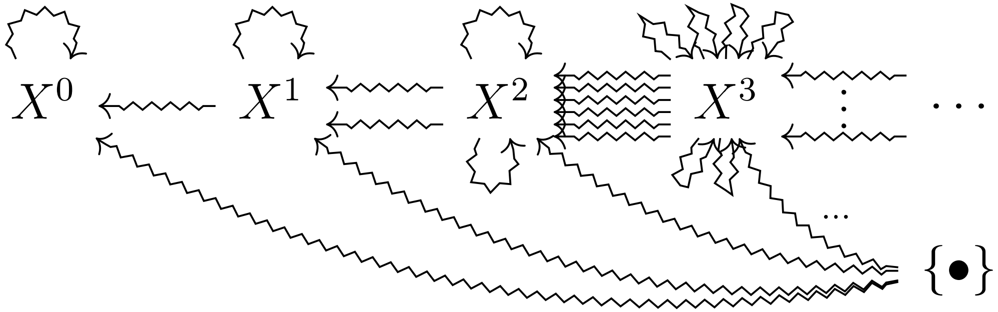
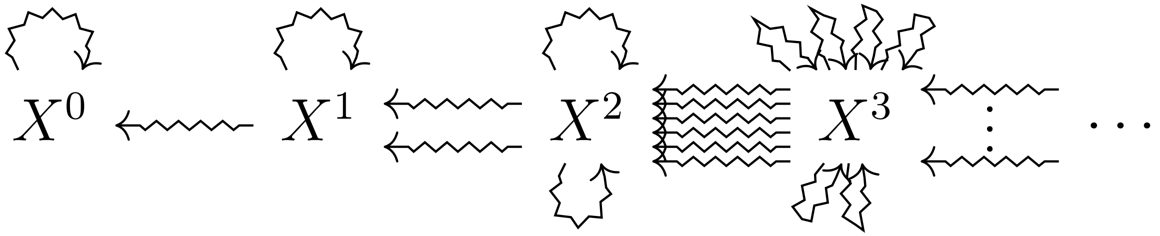
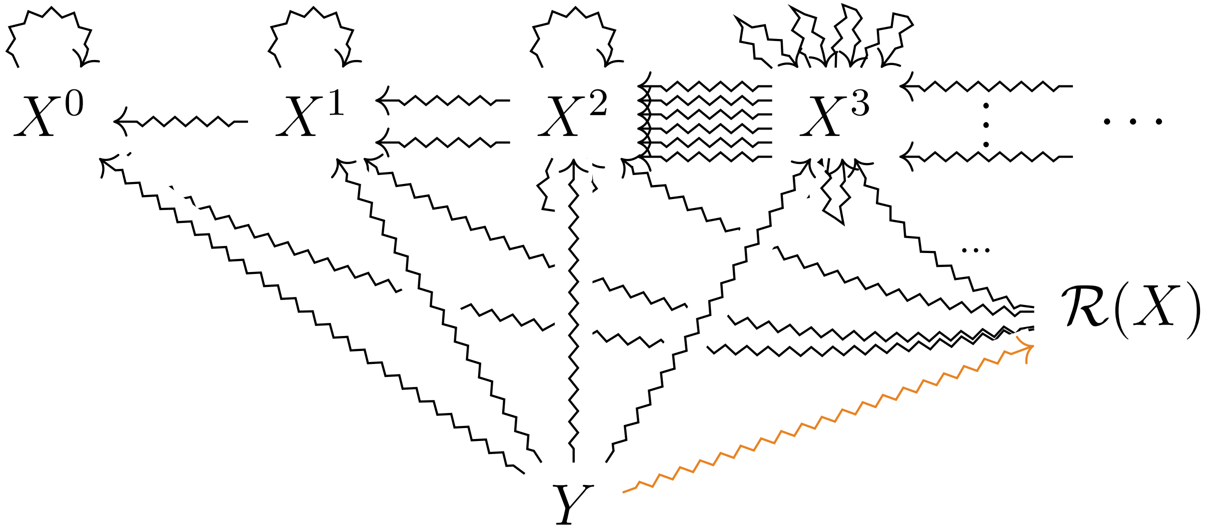


Categorical De Finetti (Kleisli Category)

Theorem: In $\mathrm{Kl}(\mathcal{R})$, $\mathcal{R}(X)$ is the limit of the diagram of permutations and projections



I.e. For any exchangeable sequence parametrized by Y:



The morphisms $\mathcal{R}(X) \rightsquigarrow X^n$ generates n independent trials from a measure: $\mu \mapsto \mu \times \cdots \times \mu$

c.f. Staton, Jacobs , 2020; Fritz, Gonda, Perrone, 2020

Exchangeability Categorically: Multisets

Let $X \in \mathbf{CH}$: How to describe an exchangeable measure?

- A set of measures μ_n on X^n for each $n \in \mathbb{N}$
- which are compatible: for $\pi_{mn}: X^m \to X^n, (\pi_{mn})_* \mu_m = \mu_n$
- and for any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, $\mu_n = (X^\sigma)_* \mu_n$

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Exchangeability Categorically: Multisets

Let $X \in \mathbf{CH}$: How to describe an exchangeable measure?

- A set of measures μ_n on $\mathcal{M}[n](X)$ for each $n \in \mathbb{N}$

Multisets: Sets with (possible) repeated elements ↔ Quotient of X^n by permutating factors
Implicit symmetry!

- which are compatible: for $\pi_{mn}: X^m \to X^n, (\pi_{mn})_* \mu_m = \mu_n$

????

Exchangeability Categorically: Multisets

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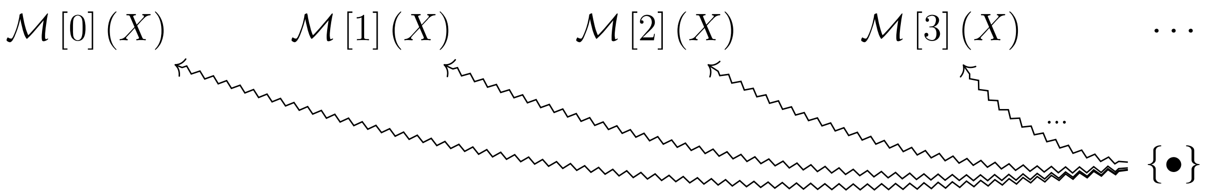
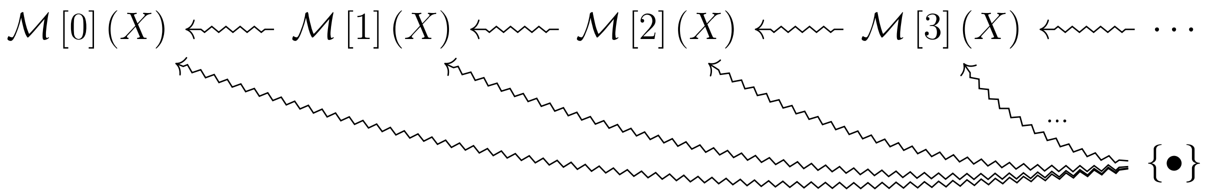


- which are compatible: for $\mathrm{DD}\colon \mathcal{M}[n+1](X) \rightsquigarrow \mathcal{M}[n](\mathrm{X})$, $(\mathrm{DD})_* \mu_m = \mu_n$

Draw-and-Delete: Randomly drop one element from the multiset.

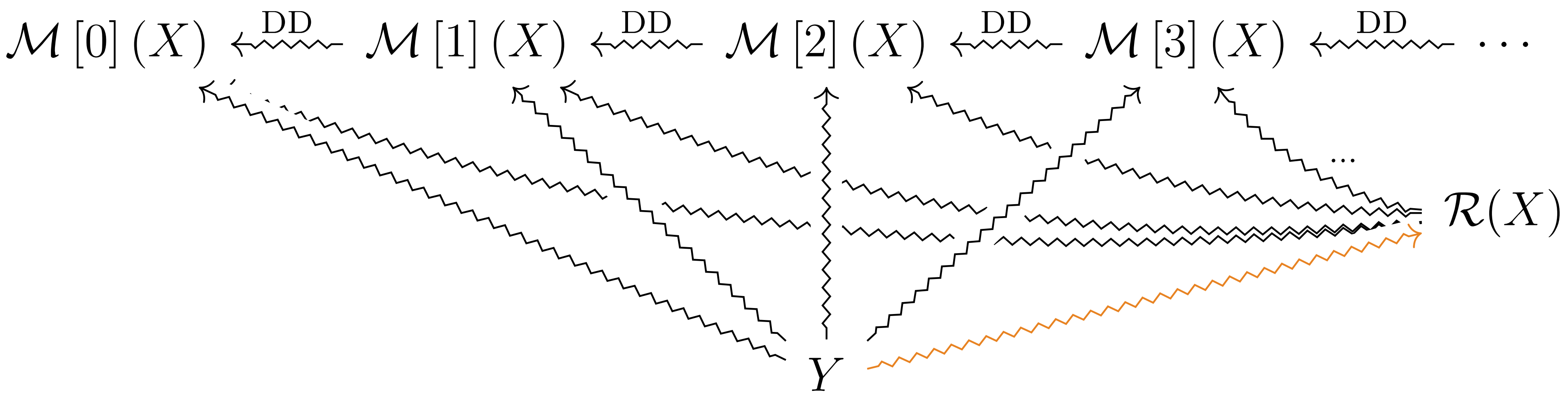


Categorical De Finetti (Multisets)

Theorem: In $\mathrm{Kl}(\mathcal{R}), \mathcal{R}(X)$ is the limit of the diagram

$$\mathcal{M}[0](X) \xleftarrow{\mathrm{DD}} \mathcal{M}[1](X) \xleftarrow{\mathrm{DD}} \mathcal{M}[2](X) \xleftarrow{\mathrm{DD}} \mathcal{M}[3](X) \xleftarrow{\mathrm{DD}} \cdots$$

I.e. For any exchangeable sequence parametrized by Y:



The morphisms $\mathcal{R}(X) \rightsquigarrow \mathcal{M}[n](X)$ generates a multiset of n independent trials from a measure.

c.f. Staton, Jacobs , 2020

Algebras of the Radon Monad

- Free algebras: $\mathcal{R}(X)$ for compact Hausdorff X
- Non-free algebras: ???

Algebras of the Radon Monad

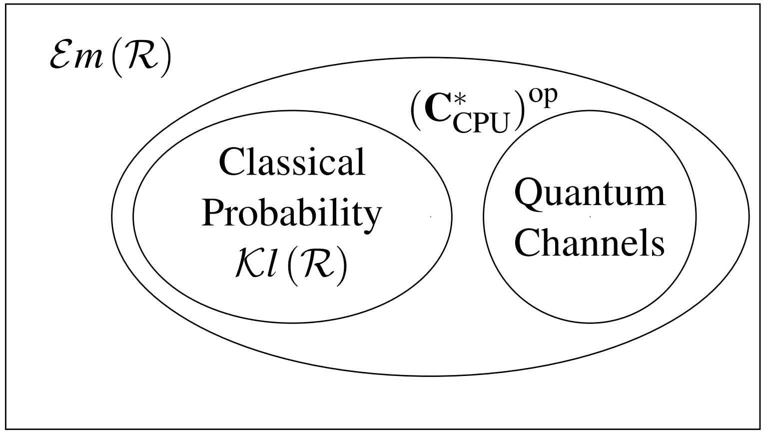
- Free algebras: $\mathcal{R}(X)$ for compact Hausdorff X
- Non-free algebras: "Quotients" of $\mathcal{R}(X)$
  - Quantum States!
- (More) formally:

$$\text{State Space Functor: Quantum Channels} \hookrightarrow \mathcal{EM}(\mathcal{R})$$

- But also we have classical probability!

$$\mathcal{K}l(\mathcal{R}) \hookrightarrow \mathcal{EM}(\mathcal{R})$$

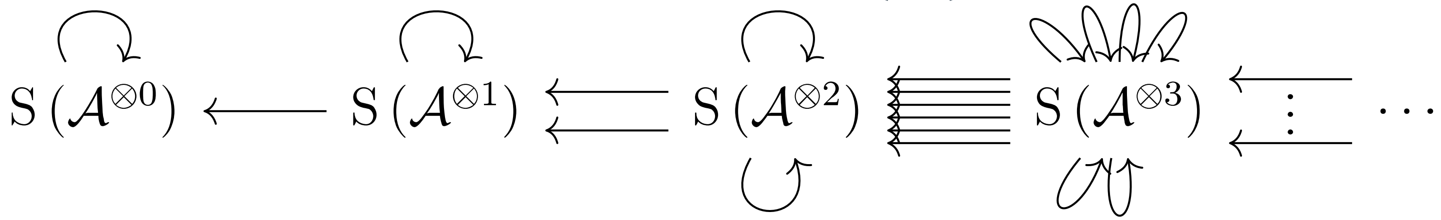
- United by C*-algebras and completely positive maps



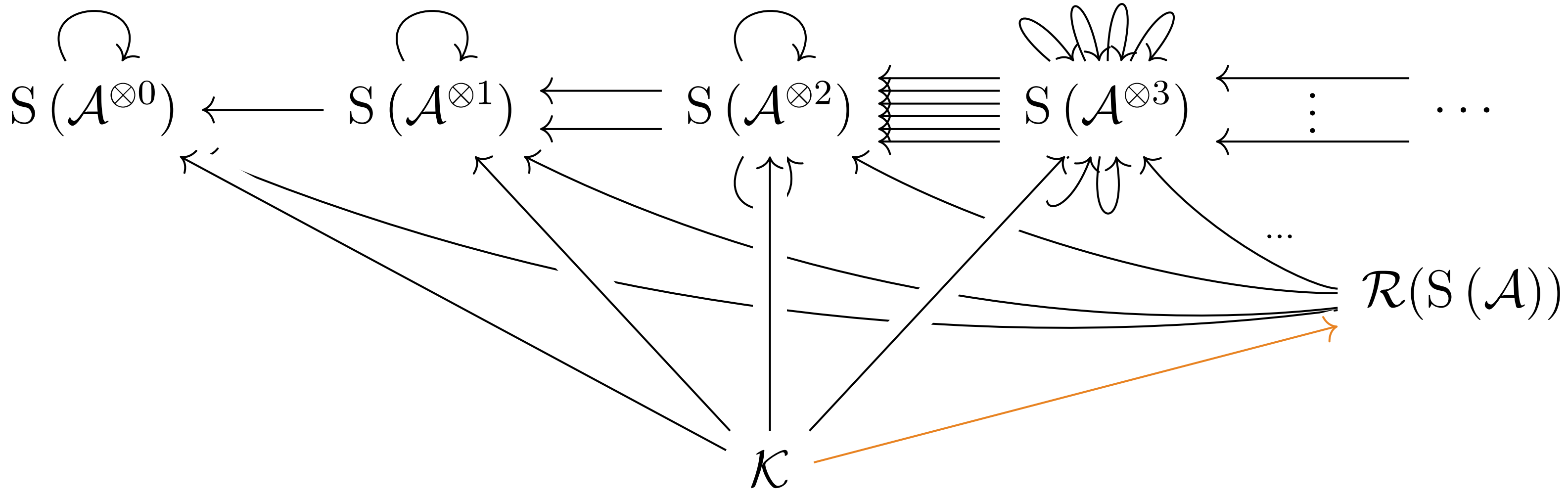
Furber, Jacobs, 2015

Categorical de Finetti: Quantum

Lemma: The space $\mathcal{R}(S(\mathcal{A}))$ is the limit of the diagram of permutations and injections in $\mathcal{EM}(\mathcal{R})$



I.e. For any exchangeable sequence parametrized by $\mathcal{K}$:



Theorem: This reflects to a similar colimit in $\mathbf{C}^*_{\mathrm{CPU}}$.

Staton, Summers, 2022; Cf. Størmer, 1969; Hudson, Moody, 1976; Caves, Fuchs, Schack, 2002

Wrapping Up

- Three Categorical De Finetti Theorems:
  - Two classical: Classifying $\mathcal{R}(X)$ as the limit of an exchangeability diagram.
    - Explicitly using the permutation maps $X^{\sigma}: X^n \to X^n$
    - Implicitly: multisets and random deletion $\mathrm{DD}: \mathcal{M}[n+1](X) \to \mathcal{M}[n](\mathrm{X})$
  - One quantum: Classifying $\mathcal{R}\big(S(\mathcal{A})\big)$ for a C*-algebra (e.g. $B(\mathcal{H})$) as limit of an exchangeability diagram.

S. Staton and N. Summers. **Quantum de Finetti Theorems as Categorical Limits, and Limits of State Spaces of C*-algebras.** To appear in Proceedings of International Conference on Quantum Physics and Logic 2022 (QPL 2022). Preprint: arxiv:2207.05832

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