Categorification of negative information using enrichment

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Double COVID combo!

- **What is negative information, and why do we care?**
  - It pops up in practical applications, e.g., infeasibility results in robot motion planning.
  - We asked: **what is the corresponding categorical notion?**
- **Idea:** represent negative information by negative arrows called **"norphisms,"** which complement the positive information of **morphisms.**
- A **nategory** is a category with some additional structure for norphisms accounting, including a compatibility relation that allows defining "**coherent subnategories**."
- Norphisms do not compose by themselves. They need a morphism as a "catalyst."

$$\frac{X \xrightarrow{f} Y \xrightarrow{g} Z}{X \xrightarrow{f \,;\, g} Z} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!\!-\!\!\bullet\, n}{\dashrightarrow} Z} \qquad \frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n \,-\!\!\bullet\!\!-\, g}{\dashrightarrow} Y}$$

  - Very **weird**, compared to the simplicity of the morphism axioms. Is this a mess? No!
- We can derive the norphism rules very elegantly using **enriched category theory.**
  - Just like a $\mathbf{P} := \langle \mathbf{Set}, \times, 1 \rangle$ -enriched category provides the data for a small category, ...
  - ... a **PN**-enriched category provides the data for a coherent subnategory.
- **Conclusions**: morphisms and norphisms are of the same substance.
  Negative information can be "categorified" using enriched category theory.

Example: robot motion planning

- **Robot motion planning:** find the optimal path between two robot configurations. Paths should avoid obstacles and have a cost (e.g., fuel required, minimum time).
  - Think of a *quasi-metric space:* (Costs are not symmetric)
    1. $d(x, y) \geq 0$
    2. $d(x, y) = 0 \iff x = y$
    3. $d(x, z) \leq d(x, y) + d(y, z)$

- As a category: objects are points in "free space," and morphisms are paths with a cost. Morphism composition concatenates the paths and "sums" the costs.

- A ***complete*** algorithm can find a **path** (if it exists) or give a **certificate of infeasibility** (if one doesn't exist).

  positive information: morphism!

  what is this, categorically?

- An ***optimal*** algorithm can find (if it exists) an **optimal solution**:
  - **a feasible path,** plus... *positive information: morphism!*
  - **a certificate of optimality**: there is no better path. *what is this, categorically?*

- Search algorithms of the A* family achieve speed using **heuristics: lower bounds for the cost between** two points.

  what is this, categorically?

Building intuition: the case of thin categories

- In a thin category, there is at most one morphism per hom-set.
- These are preorders that represent connectivity. (Motion planning without costs.)

- **We postulate these semantics:**
  - A norphisms $n : X \dashrightarrow Y$ implies that there is no morphism $f : X \to Y$
  - A morphism $f : X \to Y$ implies that there is no norphism $n : X \dashrightarrow Y$

- We find that **the norphisms rules are dual to the morphisms rules**

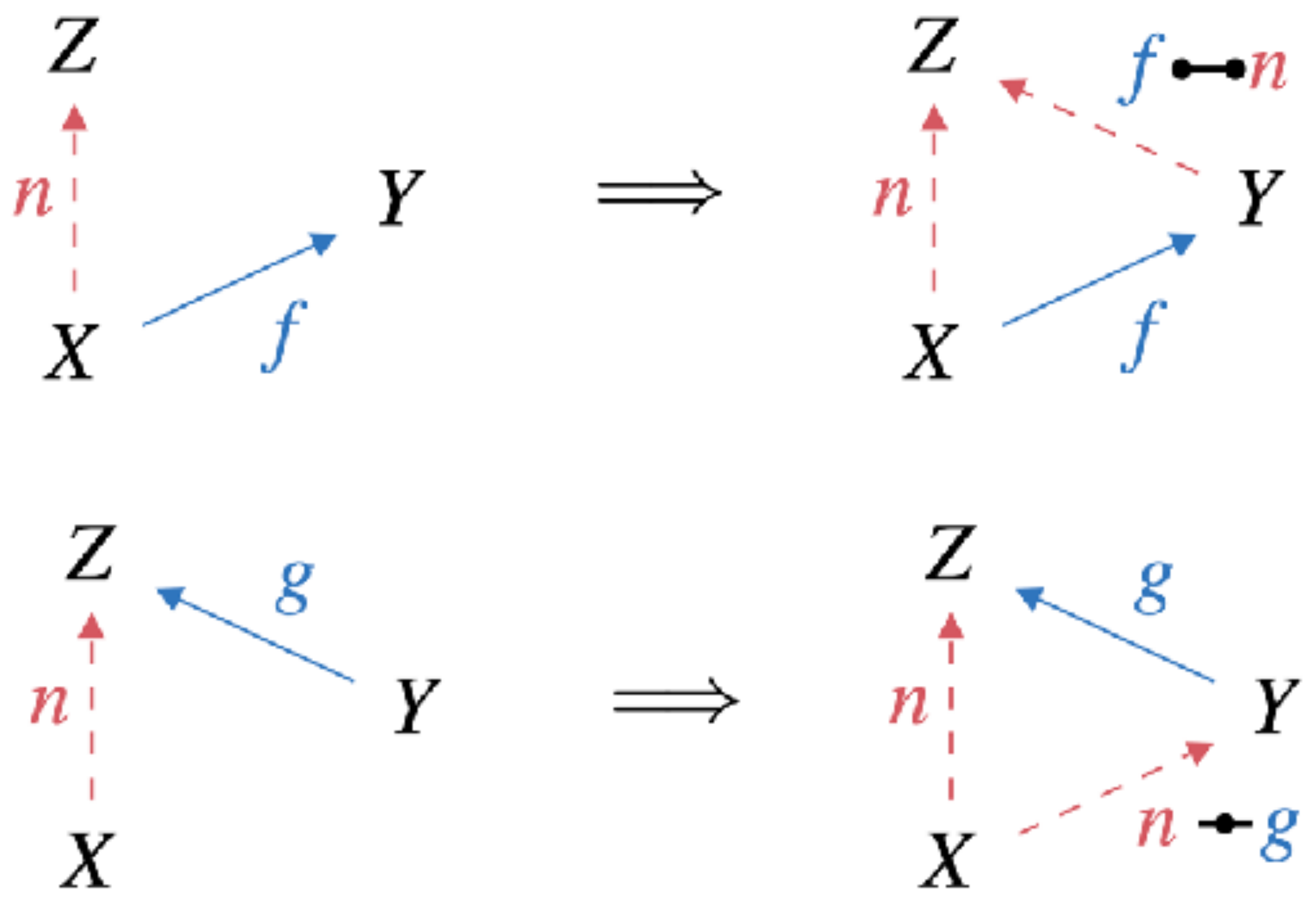
$$\frac{\top}{X \to X} \qquad \frac{f : X \to Y \quad g : Y \to Z}{(f \fatsemi g) : X \to Z}.$$

$$\frac{X \dashrightarrow X}{\bot} \qquad \frac{o : X \dashrightarrow Z \qquad Y : \mathrm{Ob}_{\mathbf{C}}}{(n : X \dashrightarrow Y) \vee (m : Y \dashrightarrow Z)}.$$

Note: nonconstructive!

Norphisms composition needs morphisms as catalysts

- We **constructively** revisit the logic to obtain **composition rules**.
- The constraint splits into **two rules** of the type morphism + norphism → norphism:



$$\frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!\!-\!\!\bullet n}{\dashrightarrow} Z}$$

$$\frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n \bullet\!- g}{\dashrightarrow} Y}.$$

- Norphism composition requires morphisms as catalysts.
- There is no norphism + norphism composition rule.

$$\frac{n : X \dashrightarrow Y \quad m : Y \dashrightarrow Z}{??? : X \dashrightarrow Z}.$$

- There is no “category of norphisms.”
- Norphisms are complementary to morphisms but obey different rules.

Nategories and coherent subnategories

Definition 1 (Nategory). A small *nategory* $\mathbf{C}$ is a small category with the following additional structure. For each pair of objects $X, Y \in \mathbf{Ob_C}$, in addition to the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, we also specify:

- A set of norphisms $\mathrm{Nom}_{\mathbf{C}}(X;Y)$. We write $n : X \dashrightarrow Y$ to say that a norphism belongs to that set.
- A *compatibility relation* between the two sets:

$$R_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X;Y) \to_{\mathbf{Rel}} \mathrm{Nom}_{\mathbf{C}}(X;Y), \tag{10}$$

where $(f R_{X,Y} n)$ means that $f : X \to Y$ is "compatible" with the norphism $n : X \dashrightarrow Y$.

Definition 2 (Subnategory). A *subnategory* $\mathbf{D}$ of $\mathbf{C}$ is a nategory $\mathbf{D}$ that is a subcategory of $\mathbf{C}$ in the usual sense, and for which $\mathrm{Nom}_{\mathbf{D}}(X;Y) \subseteq \mathrm{Nom}_{\mathbf{C}}(X;Y)$.

Definition 3 (Coherent subnategory). A subnategory $\mathbf{D}$ of $\mathbf{C}$ is *coherent* if all morphisms and norphisms are compatible:

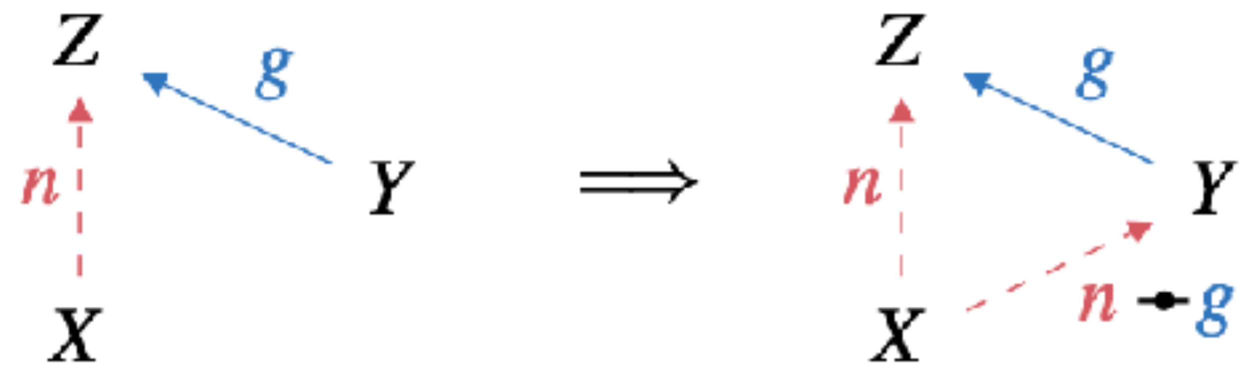
$$\frac{f : \mathrm{Hom}_{\mathbf{D}}(X;Y) \quad n : \mathrm{Nom}_{\mathbf{D}}(X;Y)}{f (R_{X,Y}) n}. \tag{11}$$

- **Interpretation as a generalization of subcategories**:
  - Think of a subcategory as a "coherent view" of a category, in the sense that it is a selection of morphisms closed to composition.
  - We are generalizing the notion of subcategory by adding norphisms (that must be compatible with the present morphisms).
- Thinking of **coherent subnategory as states of information** allows distinguishing
  - **absence of evidence** *(e.g., an empty subcategory)*

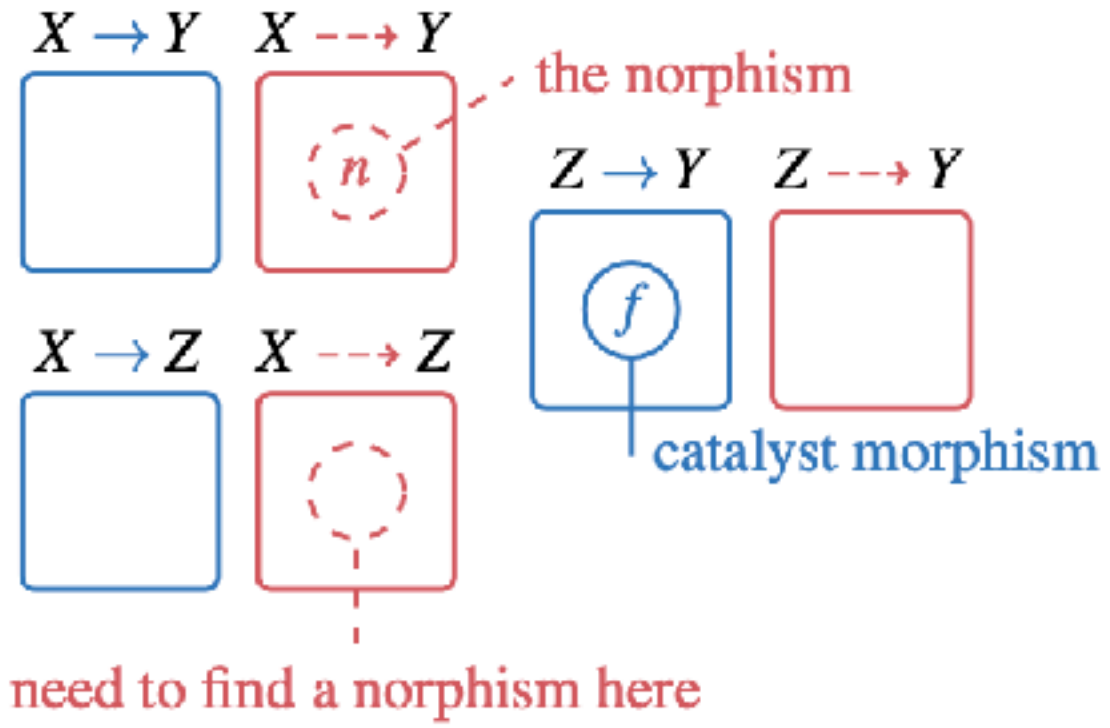
    vs
  - **evidence of absence** *(e.g., enough norphisms to negate the existence of all morphisms)*

Deriving norphism composition rules

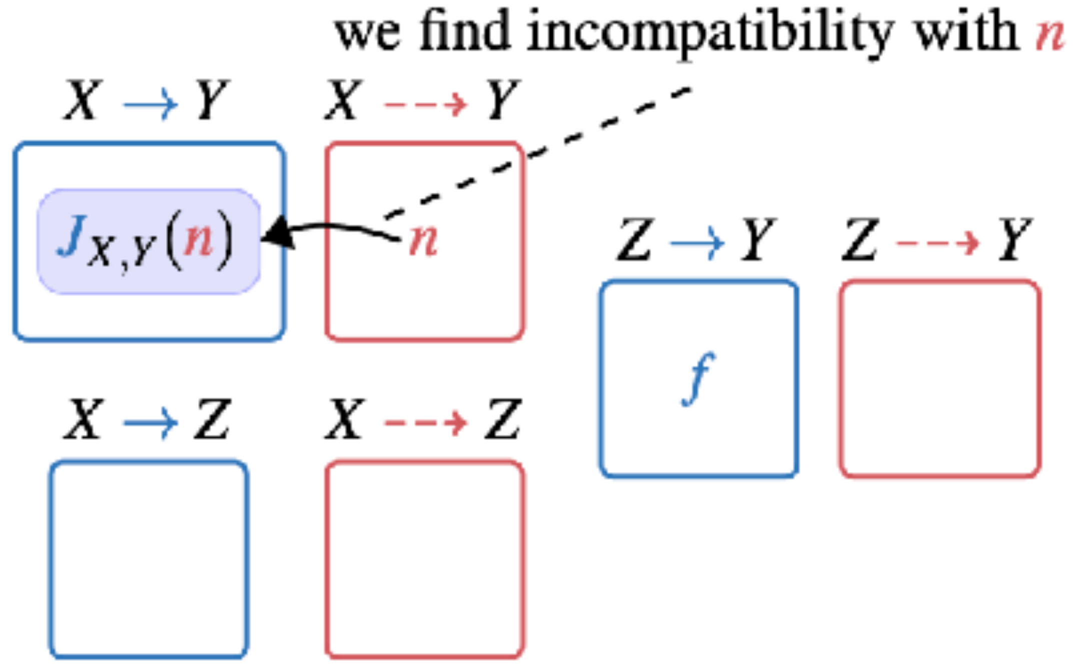
- From this nategory structure we can define the two composition operators.



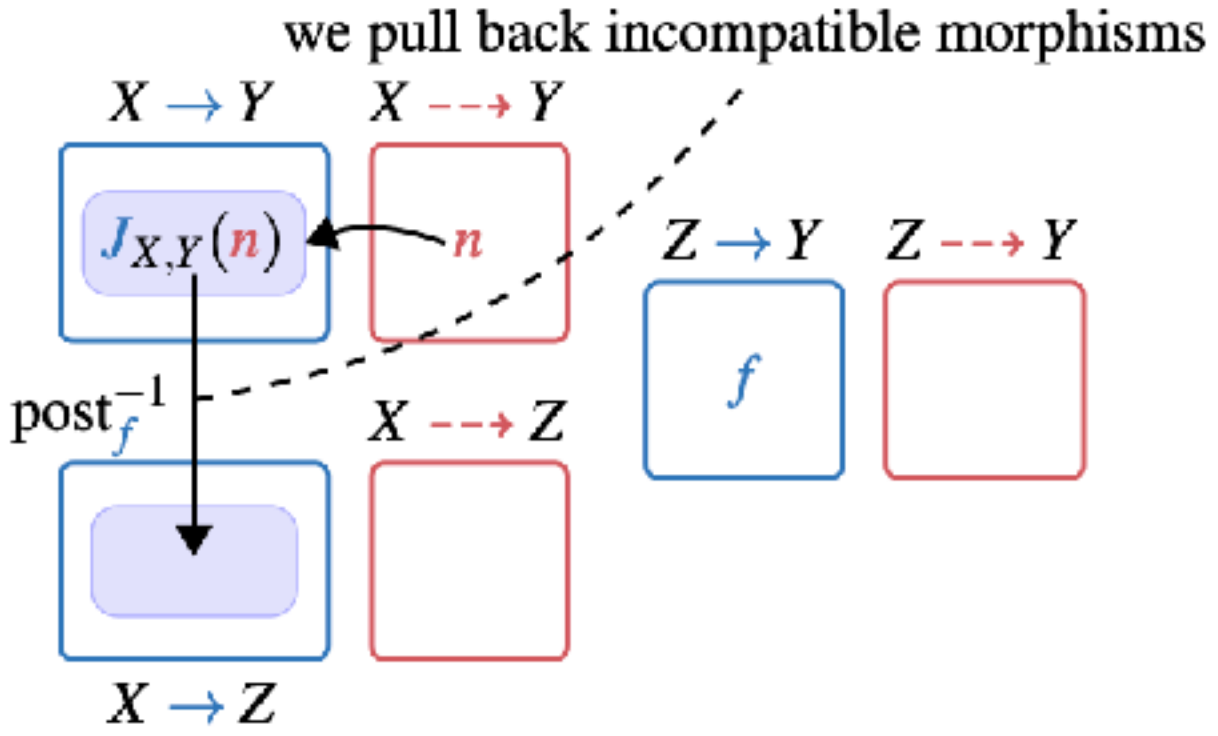
$$\frac{X \overset{n}{\dashrightarrow} Z \overset{g}{\leftarrow} Y}{X \overset{n \multimap g}{\dashrightarrow} Y}\,.$$



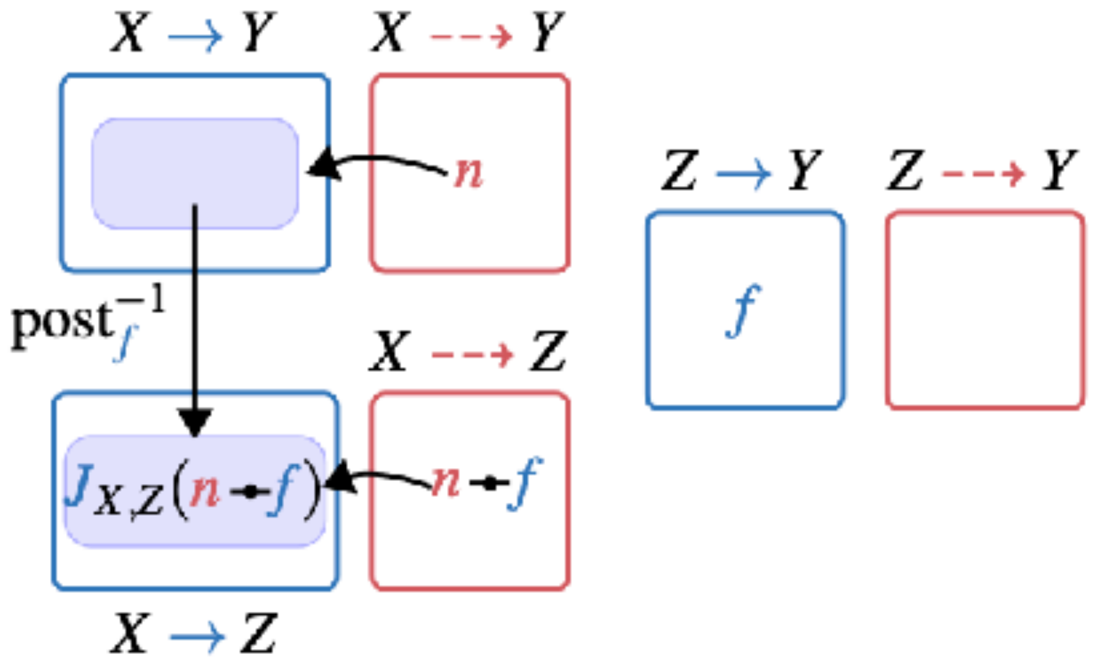
(a) Starting situation.



(b) Find incompatibility with n.



(c) Pulling back incompatible morphisms.



(d) We have now found $n \multimap f$.

Deriving norphism composition rules

- From this nategory structure we can define the two composition operators.

$$
\begin{aligned}
I_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Nom}_{\mathbf{C}}(X;Y)),\\
f &\mapsto \{n \in \mathrm{Nom}_{\mathbf{C}}(X;Y) : \neg f R_{X,Y} n\},
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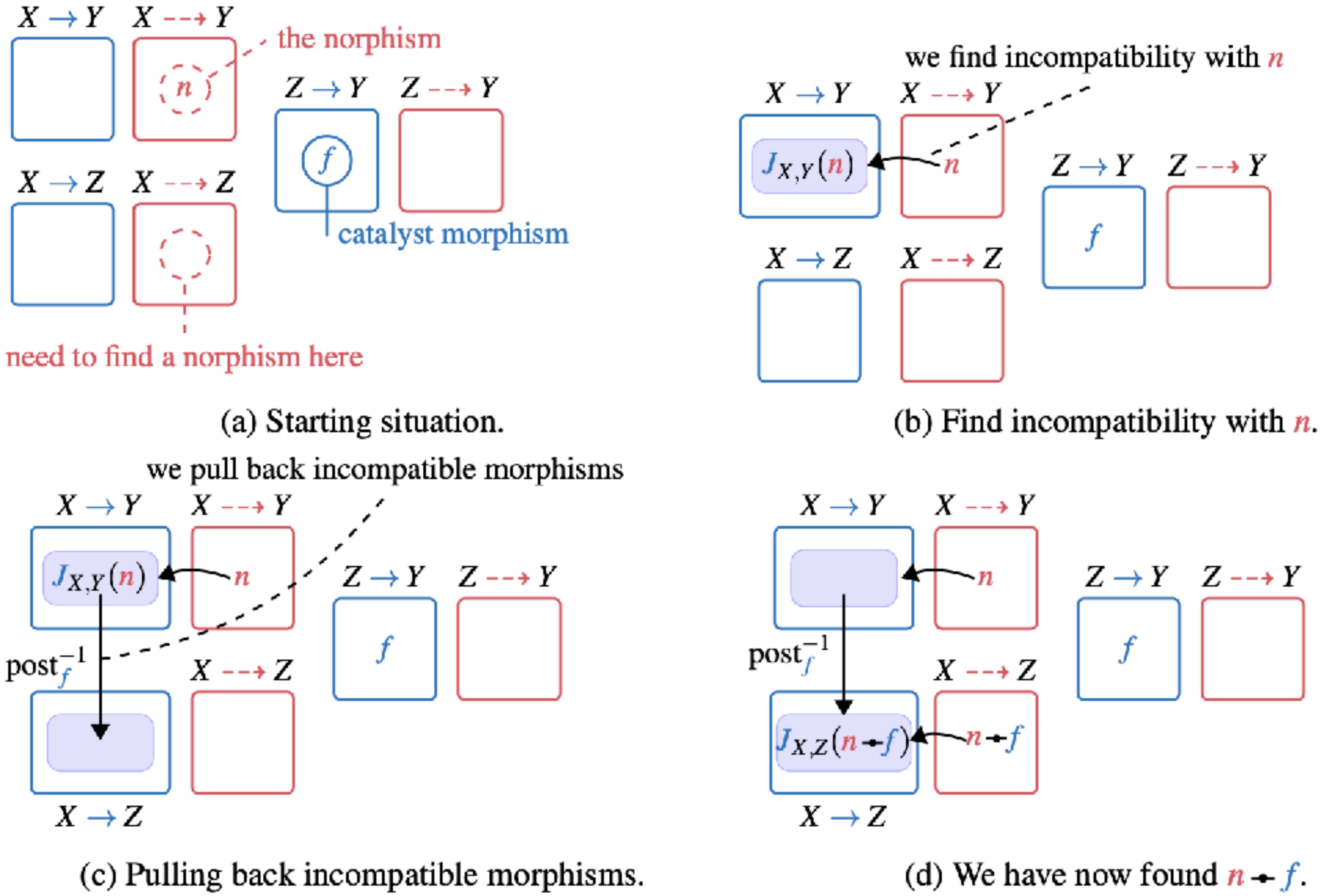


(a) Starting situation.

(b) Find incompatibility with n.

(c) Pulling back incompatible morphisms.

(d) We have now found $n \multimap f$.

$$
\frac{X \overset{n}{\dashrightarrow} Z \overset{g}{\leftarrow} Y}{X \overset{n \multimap g}{\dashrightarrow} Y}\ .
$$

Example: hiking on the Swiss mountains

Definition 5 (Berg). Let $h\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a C^1 function, describing the elevation of a mountain. The set with elements $\langle a, b, h(a,b)\rangle$ is a manifold $\mathbb{M}$ that is embedded in $\mathbb{R}^3$. Let $\sigma = [\sigma_L, \sigma_U] \subset \mathbb{R}$ be a closed interval of real numbers. The category $\mathbf{Berg}_{h,\sigma}$ is specified as follows:

1. An object X is a pair $\langle \mathbf{p}, \mathbf{v}\rangle \in \mathscr{T}\mathbb{M}$, where $\mathbf{p} = \langle \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\rangle$ is the position, $\mathbf{v}$ is the velocity, and $\mathscr{T}\mathbb{M}$ is the tangent bundle of the manifold.
2. Morphisms are C^1 paths on the manifold.At each point of a path we define the *steepness* as:

$$s(\langle \mathbf{p}, \mathbf{v}\rangle) := \mathbf{v}_z / \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2}. \tag{18}$$

We choose as morphisms only the paths that have the steepness values contained in the interval σ:

$$\mathrm{Hom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) = \{f \text{ is a } C^1 \text{ path from } X \text{ to } Y \text{ and } s(f) \subseteq \sigma\}, \tag{19}$$

3. Morphism composition is given by concatenation of paths.
4. Given any object, the identity morphism is the trivial self path with only one point.

- We take **norphisms in Berg** to be lower bounds on the path length:

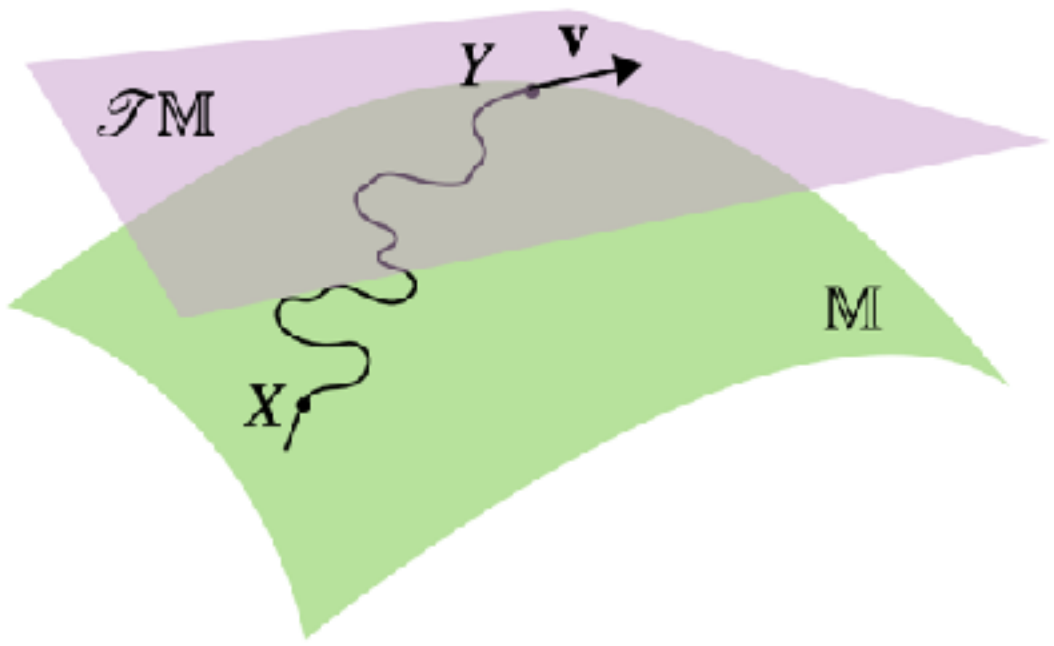
$$\mathrm{Nom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

- The **compatibility condition** says that a norphism and a morphism are compatible if the lower bound is not violated.

$$\frac{f R_{X,Y} n}{\mathrm{length}(f) \geq n}$$

- An **optimal path is a pair** of morphism and a norphism:

$$\frac{f : X \to Y \quad \mathrm{length}(f) : X \dashrightarrow Y}{f \text{ is optimal}}$$



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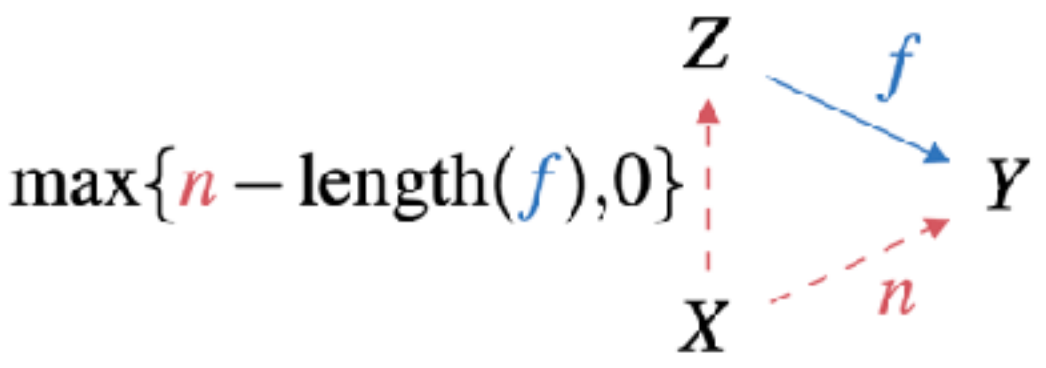

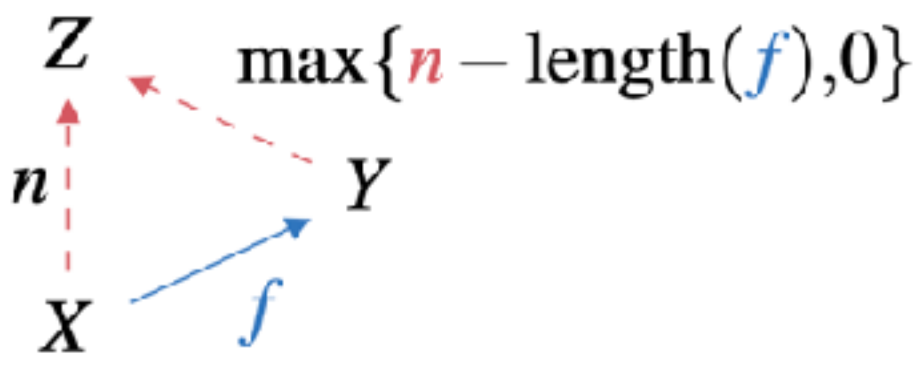

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- **Some norphisms axioms schemas** that we could use in **Berg.**
  - The length of a path cannot be lower than the distance in 3D: $\|\mathbf{p}^1 - \mathbf{p}^2\| \colon \langle \mathbf{p}^1, \mathbf{v}^1\rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2\rangle$
  - The length of a path cannot be lower than than the geodesic distance:

$$d_{\mathbb{M}}(\mathbf{p}^1, \mathbf{p}^2) \colon \langle \mathbf{p}^1, \mathbf{v}^1\rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2\rangle$$

  - Moreover, the following bounds hold due to the constraint on inclination:

$$\mathbf{p}_z^1 - \mathbf{p}_z^2 < 0$$

$$|\mathbf{p}_z^1 - \mathbf{p}_z^2| / \sigma_U \colon \langle \mathbf{p}^1, \mathbf{v}^1\rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2\rangle$$

$$\mathbf{p}_z^1 - \mathbf{p}_z^2 > 0$$

$$|\mathbf{p}_z^1 - \mathbf{p}_z^2| / \sigma_L \colon \langle \mathbf{p}^1, \mathbf{v}^1\rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2\rangle$$

Enlightenment by Enrichment

Definition 7 (Enriched category). Let $\langle \mathbf{V}, \otimes, \mathbf{1}, as, lu, ru \rangle$ be a monoidal category, where as is the associator, lu is the left unitor, and ru is the right unitor.

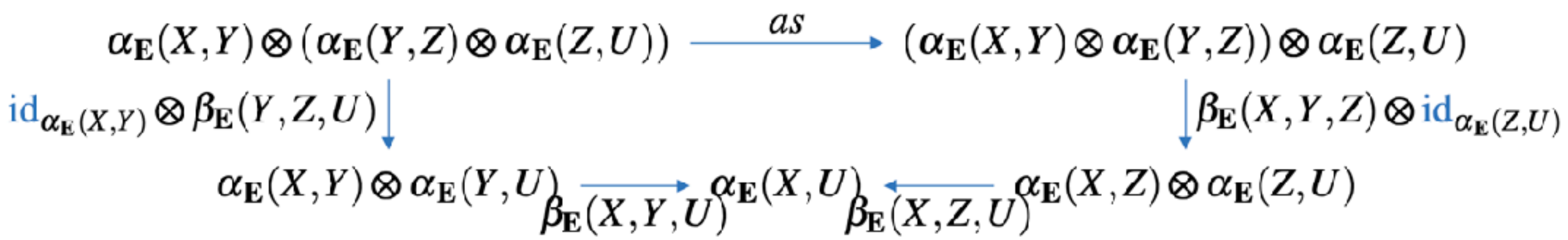
A $\mathbf{V}$-enriched category $\mathbf{E}$ is given by a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$, where

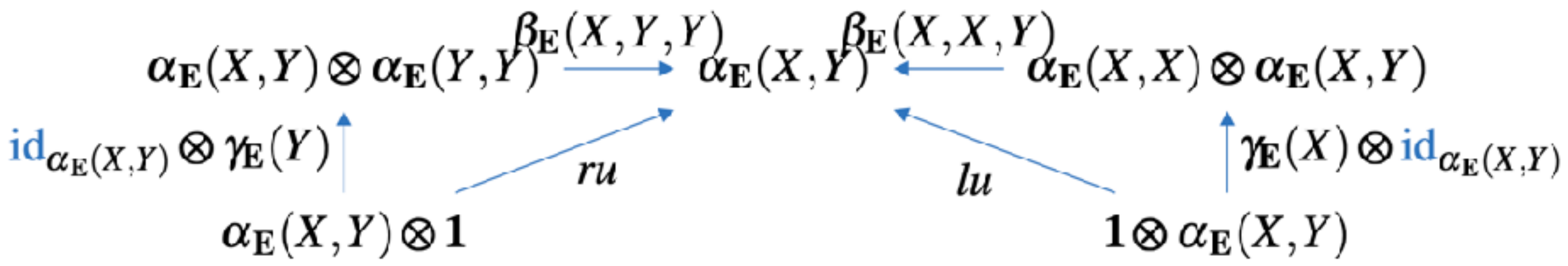
1. $\mathrm{Ob}_{\mathbf{E}}$ is a set of "objects".
2. $\alpha_{\mathbf{E}}$ is a function such that, for all pairs of objects $X, Y \in \mathrm{Ob}_{\mathbf{E}}$, the value $\alpha_{\mathbf{E}}(X,Y)$ is an object of $\mathbf{V}$.
3. $\beta_{\mathbf{E}}$ is a function such that, for all $X, Y, Z \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism $\beta_{\mathbf{E}}(X,Y,Z)$ of $\mathbf{V}$, called *composition morphism*:

$$\beta_{\mathbf{E}}(X,Y,Z) \colon \alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,Z) \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,Z). \tag{32}$$

4. $\gamma_{\mathbf{E}}$ is a function such that, for each $X \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism of $\mathbf{V}$:

$$\gamma_{\mathbf{E}}(X) \colon \mathbf{1} \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,X). \tag{33}$$





The category PN

- **Idea**: objects and morphisms are **dependent pairs** with a positive and a negative part.
  - The positive part is a copy of $\mathbf{P} := \langle \mathbf{Set}, \times, 1 \rangle$
  - The negative part is intertwined with **P** and cannot be factorized.
    However, for intuition, think about **N** being a dependent version of $\mathbf{N}_0 := \langle \mathbf{Set}^{\mathrm{op}}, +, \emptyset \rangle$

Definition 9 (Category **PN**). The category **PN** is defined as follows.

1. The objects of **PN** are dependent pairs $\langle H, m \colon H \to \mathsf{Pow}(N) \rangle$, where H, N are sets, and m is a map that associates to an element of H a subset of N.
2. A morphism $f \colon \langle H_1, m_1 \rangle \to \langle H_2, m_2 \rangle$ is a pair of functions $\langle \varphi, \psi \rangle$ where
$$\begin{aligned} \varphi &\colon H_1 \to H_2, \\ \psi &\colon (h_1 : H_1) \to (m_2(\varphi(h_1)) \to m_1(h_1)). \end{aligned} \tag{37}$$
3. Given morphisms $f \colon \langle H_1, m_1 \rangle \to \langle H_2, m_2 \rangle$ and $g \colon \langle H_2, m_2 \rangle \to \langle H_3, m_3 \rangle$, their composition is a morphism $f \mathbin{;} g$, where
$$\begin{aligned} \varphi_{f;g} &= \varphi_f \mathbin{;} \varphi_g, \\ \psi_{f;g}(h_1) &= \psi_g(\varphi_f(h_1)) \mathbin{;} \psi_f(h_1). \end{aligned} \tag{38}$$
4. An identity for an object $\langle H, m \rangle$ is given by
$$\varphi = \mathrm{id}_H, \quad \psi(h) = \mathrm{id}_{m(h)}, \tag{39}$$
where id_H is the identity function on the set H and $\mathrm{id}_{m(h)}$ is the identity function on the set $m(h)$.

- A reviewer asked: Is **PN** a submonoidal category of **Poly**? Maybe, with some changes.

  *I am also becoming **Poly**amorous!*

Defining a monoidal structure on PN

- A preliminary definition to compose the "negative part" of the PN morphisms:

Definition 10 ("Δ"). Given two maps $m_1 : H_1 \to \mathsf{Pow}(N_1)$ and $m_2 : H_2 \to \mathsf{Pow}(N_2)$, we define

$$\begin{aligned}(m_1 \Delta m_2) : H_1 \times H_2 &\to \mathsf{Pow}(N_1 + N_2),\\ \langle h_1, h_2 \rangle &\mapsto \mathsf{in}_1(m_1(h_1)) \cup \mathsf{in}_2(m_2(h_2)),\end{aligned}$$

where $\mathsf{in}_1, \mathsf{in}_2$ are the injections in the disjoint union lifted to sets.

  - The operation has this identity: $\begin{aligned}\mathrm{id}_\Delta : 1 &\to \mathsf{Pow}(\emptyset),\\ \bullet &\mapsto \emptyset.\end{aligned}$

- **Definition of a monoidal structure on PN:**

Lemma 11. $\langle \mathbf{PN}, \otimes_{\mathbf{PN}}, \langle 1, \mathrm{id}_\Delta \rangle \rangle$ is a monoidal category, defining the product of two objects as

$$\langle H_1, m_1 \rangle \otimes_{\mathbf{PN}} \langle H_2, m_2 \rangle := \langle H_1 \times H_2, m_1 \Delta m_2 \rangle,$$

and the product of two morphisms $f : \langle H_1, m_1 \rangle \to \langle K_1, l_1 \rangle$, $g : \langle H_2, m_2 \rangle \to \langle K_2, l_2 \rangle$

$$f \otimes_{\mathbf{PN}} g : \langle H_1 \times H_2, m_1 \Delta m_2 \rangle \to \langle K_1 \times K_2, l_1 \Delta l_2 \rangle$$

as the morphism defined by the two functions $\varphi_{f \otimes_{\mathbf{PN}} g}$ and $\psi_{f \otimes_{\mathbf{PN}} g}$ defined as

$$\begin{aligned}\varphi_{f \otimes_{\mathbf{PN}} g} &= \varphi_f \times \varphi_g,\\ \psi_{f \otimes_{\mathbf{PN}} g} &: (\langle h_1, h_2 \rangle : H_1 \times H_2) \to \psi_f(h_1) + \psi_g(h_2),\end{aligned}$$

where $\times$ is the product of functions and $+$ is the direct sum of functions.

Enrichment in PN describes coherent subnategories

- Recall the following fact about enrichment in $\mathbf{P} := \langle \mathbf{Set}, \times, 1 \rangle$:

Lemma 8. A category enriched in **P** gives the data necessary to define a small category.

Proof. We show one direction. Suppose that we are given a **P**-enriched category as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$. We can define a small category **C** as follows:

- Set $\mathrm{Ob}_{\mathbf{C}} := \mathrm{Ob}_{\mathbf{E}}$.
- For each $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, let $\mathrm{Hom}_{\mathbf{C}}(X;Y) := \alpha_{\mathbf{E}}(X,Y)$.
- For each $X, Y, Z \in \mathrm{Ob}_{\mathbf{C}}$, we know a function

$$\beta_{\mathbf{E}}(X,Y,Z) : \mathrm{Hom}_{\mathbf{C}}(X;Y) \otimes \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X;Z). \tag{35}$$

  The diagrams constraints imply that this function is associative.
  Therefore, we use it to define morphism composition in **C**, setting $\fatsemi_{X,Y,Z} := \beta_{\mathbf{E}}(X,Y,Z)$.
- For each $X \in \mathrm{Ob}_{\mathbf{C}}$ we know a function $\gamma_{\mathbf{E}}(X) : 1 \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X;X)$ that selects a morphism.
  The diagrams constraints imply that such morphism satisfies unitality with respect to $\fatsemi_{X,Y,Z}$.
  Therefore, we can use it to define the identity at each object:

$$\mathrm{id}_X := \gamma_{\mathbf{E}}(X)(\bullet). \tag{36}$$

- **We can prove an analogous result for PN:**

Proposition 12. A **PN**-enriched category provides the data necessary to specify a coherent subnategory.

- The "**P**" part recovers the category structure (positive information), as in the traditional construction.
- The "**N**" part recovers the nategory structure (nom-sets, compatibility relation, norphism composition).

Highlight from the proof

Proposition 12. A **PN**-enriched category provides the data necessary to specify a coherent subnategory.

- Define the function I that gives the norphisms that are incompatible with a particular morphism:

$$\begin{aligned} I_{X,Y}\colon \mathrm{Hom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Nom}_{\mathbf{C}}(X;Y)),\\ f &\mapsto \{n \in \mathrm{Nom}_{\mathbf{C}}(X;Y) \colon \neg f R_{X,Y} n\} \end{aligned}$$

- In the proof, for all triples of objects X, Y, Z, we construct the dependent function

$$\psi\colon (\langle f, g\rangle \colon \mathrm{Hom}(X;Y) \times \mathrm{Hom}(Y;Z)) \to (I_{X,Z}(f \mathbin{⨾} g) \to (\mathsf{in}_1(I_{X,Y}(f)) \cup \mathsf{in}_2(I_{Y,Z}(g)))).$$

- Reading this using the "**propositions as types**" interpretation gives **the logic for norphisms and the norphism composition rules.**

For all f, g $\longrightarrow$ $I_{X,Z}(f \mathbin{⨾} g)$ —nonempty?→

The composite morphism $f \mathbin{⨾} g$ is **not** allowed in this subnategory.

The elements of $I_{X,Z}(f \mathbin{⨾} g)$ are the norphisms that contradict $f \mathbin{⨾} g$.
For each norphism n, we can evaluate the function

$$I_{X,Z}(f \mathbin{⨾} g) \to (\mathsf{in}_1(I_{X,Y}(f)) \cup \mathsf{in}_2(I_{Y,Z}(g)))$$

$I_{X,Y}(f)$ empty ↓

If f is not denied, we obtain an element of $I_{Y,Z}(g)$

$$\frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!\!-\!\!\bullet n}{\dashrightarrow} Z}$$

$I_{Y,Z}(g)$ empty ↓

If g is not denied, we obtain an element of $I_{X,Y}(f)$

$$\frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n \bullet\!\!-\!\!\bullet g}{\dashrightarrow} Y}$$

empty? ↓

The composite morphism $f \mathbin{⨾} g$ is allowed in this subnategory.
There are no norphisms that deny it.

Conclusions and future work

- **Negative information can be categorified** using negative arrows (norphisms).
  - (as opposed to using some logic on top of category theory...)
- Norphisms behave fundamentally differently than morphisms.
  They compose using morphisms as catalysts.

$$\frac{X \xrightarrow{f} Y \xrightarrow{g} Z}{X \xrightarrow{f \fatsemi g} Z} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!\!-\!\!\bullet n}{\dashrightarrow} Z} \quad \frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n -\!\!\bullet\!\!- g}{\dashrightarrow} Y}$$

- **"Nategories" generalize categories** to account for the norphism machinery.
- **"Coherent subnategories" generalize subcategories** by adding a selection of norphisms that must be compatible with the selection of morphisms. **A coherent state of information**.
- We can derive the norphism rules very elegantly using **enriched category theory.**
  - Just like a **Set**-enriched category provides the data for a small category, ...
  - ... a **PN**-enriched category provides the data for a coherent subnategory.

- **Future work**
  - Surveying natural norphism structures in the wild.
  - Explore more the idea of algorithms producing both positive and negative information.
  - Is **PN** a submonoidal category of **Poly**? (asked a reviewer) Maybe, with some changes.
  - Generalization to higher-level concepts. What would a "nunctor" be?