On Ramsey Theory, Category Theory and Entropy

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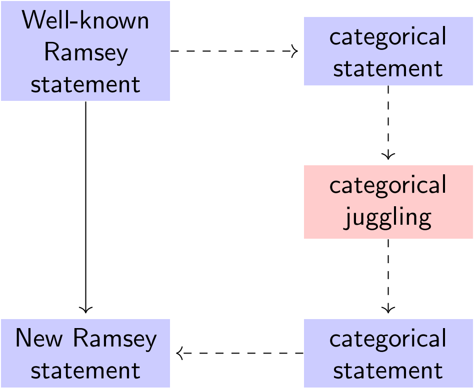
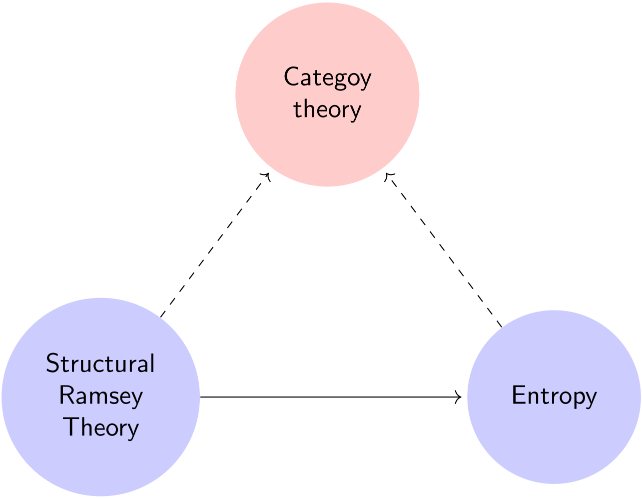
Categorical Ramsey Theory

- Duality Principle facilitates reasoning about dual Ramsey phenomena;
- piggyback proof strategies:



Categorical Ramsey Theory



Entropy of combinatorial structures

Example. How "unexpected" are the following graphs?

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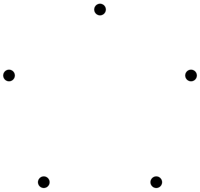

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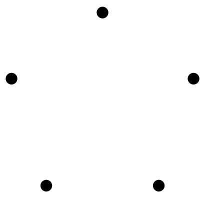

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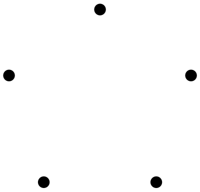

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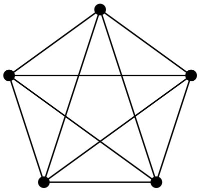

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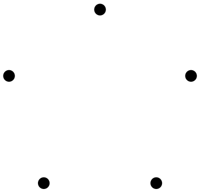

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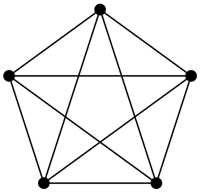

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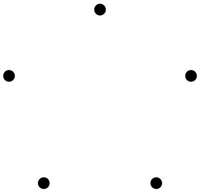

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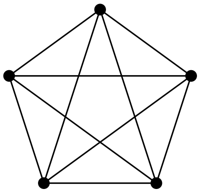

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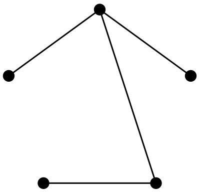

somewhat

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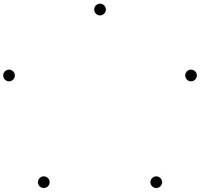

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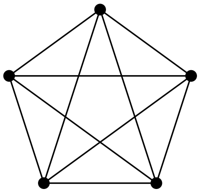

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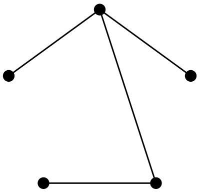

somewhat

$|\mathrm{Aut}(G)| = 5!$

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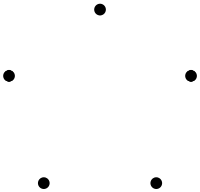

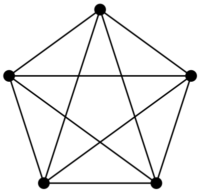

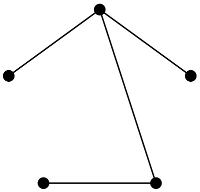

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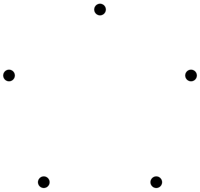

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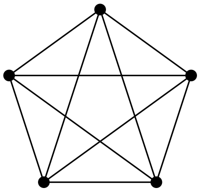

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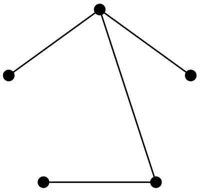

somewhat

$|\text{Aut}(G)| = 2$

Entropy of combinatorial structures

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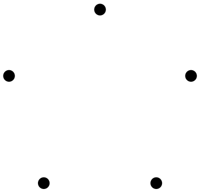

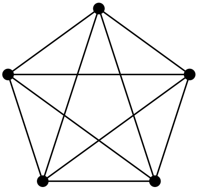

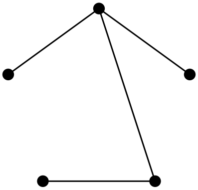

not at all — not at all — somewhat

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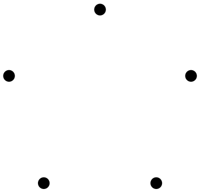

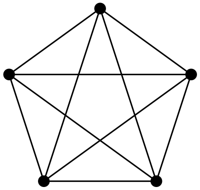

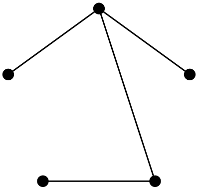

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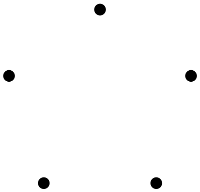

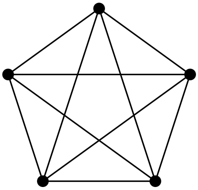

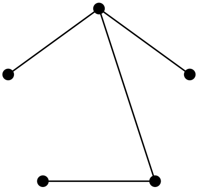

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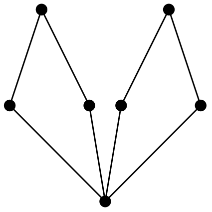

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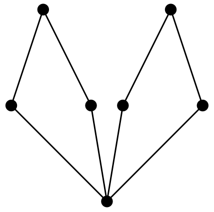

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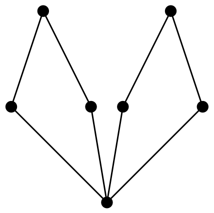

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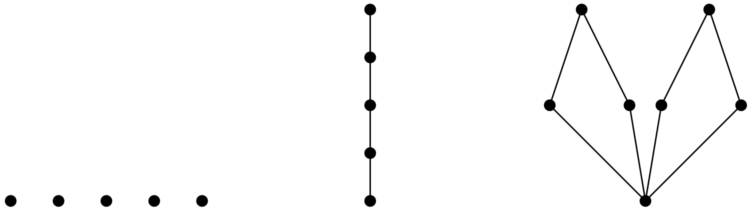

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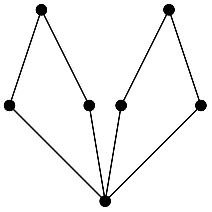

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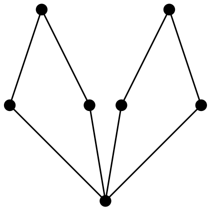

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$|\mathrm{Aut}(P)| = 5!$ | $|\mathrm{Aut}(P)| = 1$ | $|\mathrm{Aut}(P)| = 8$

$\log_2 \frac{\mathrm{linext}(P)}{|\mathrm{Aut}(P)|} = 0$

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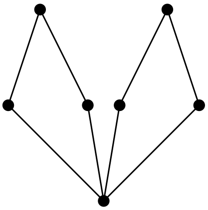

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		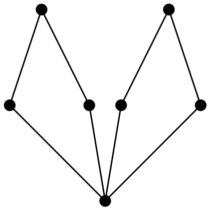
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$\log_2 \frac{\mathrm{linext}(P)}{\lvert \mathrm{Aut}(P) \rvert} = 0$	$\log_2 \frac{\mathrm{linext}(P)}{\lvert \mathrm{Aut}(P) \rvert} = 0$	$\log_2 \frac{\mathrm{linext}(P)}{\lvert \mathrm{Aut}(P) \rvert} \approx 3.32$

Entropy of combinatorial structures

NB.

- $\frac{n!}{|\mathrm{Aut}(G)|} =$ small structural Ramsey degree of a graph G
- $\frac{\mathrm{linext}(P)}{|\mathrm{Aut}(P)|} =$ small structural Ramsey degree of a poset P

Entropy of combinatorial structures

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$$H(A) = \log_2 \tilde{t}(A)$$

ALAS, NO!

✗ $A \preccurlyeq B \Rightarrow H(A) \leqslant H(B)$ for some “natural” ordering $\preccurlyeq$;

Expectations

- $H(A) \leqslant \log \delta(A)$, where $\delta(A)$ measures "diversity";
- $A \cong B \Rightarrow H(A) = H(B)$;
- $H(A) = 0$ iff A is "simple";
- $A \preccurlyeq B \Rightarrow H(A) \leqslant H(B)$ for some "natural" ordering $\preccurlyeq$;
- $H(A, B) = H(A) + H(B)$.

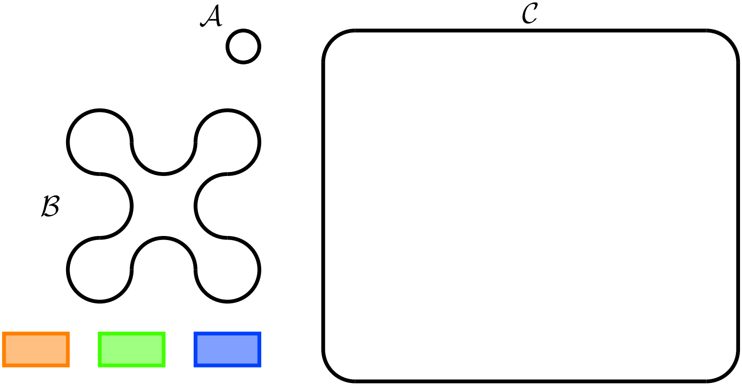
Structural Ramsey Theory

Deep structural property developed in the 1970's by Erdős, Graham, Leeb, Rothschild, Rödl, Nešetřil and many more.

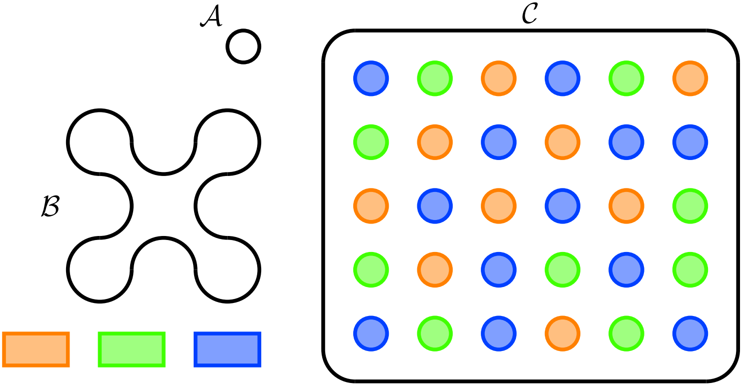
Definition.
A class $\mathbf{K}$ of finite structures has the Ramsey property if:

> for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ and all $k \geqslant 2$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$.

$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$

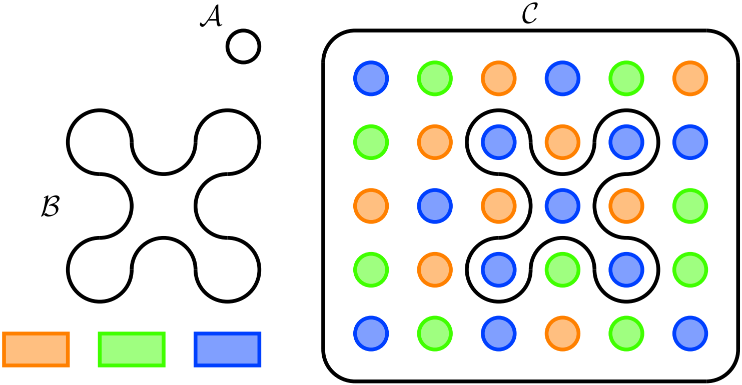


$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$



for every coloring $\chi : \binom{\mathcal{C}}{\mathcal{A}} \to k$

$\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_{k}$



there is a $\tilde{\mathcal{B}} \in \binom{\mathcal{C}}{\mathcal{B}}$ such that $\left|\chi\left(\binom{\tilde{\mathcal{B}}}{\mathcal{A}}\right)\right| = 1.$

SURPRISE! NO combinatorially interesting class of finite structures has the Ramsey property!

Structural Ramsey Theory

SURPRISE! NO combinatorially interesting class of finite structures has the Ramsey property!

Two approaches to rectify the injustice:

1. early 1970's: add more structure
   $\rightsquigarrow$ *precompact Ramsey expansions*, Nguyen Van Thé 2013
2. late 1990's: relax the Ramsey property
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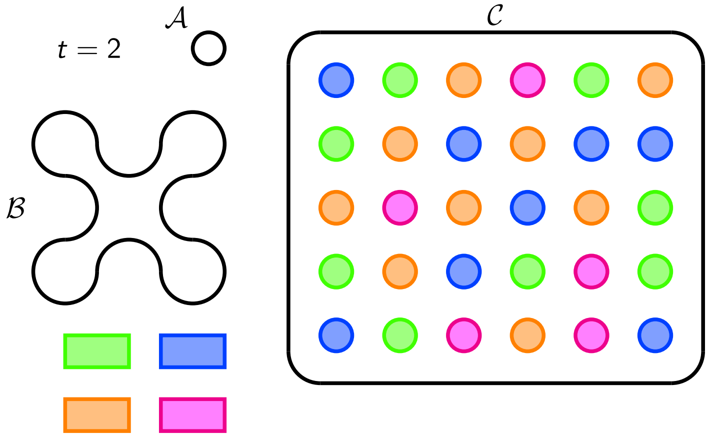
2016 Andy Zucker: the two approaches are equivalent!

- An amalgamation class of finite structures has finite Ramsey degrees iff it has a precompact Ramsey expansion.

Structural Ramsey Theory

Relaxing the property.



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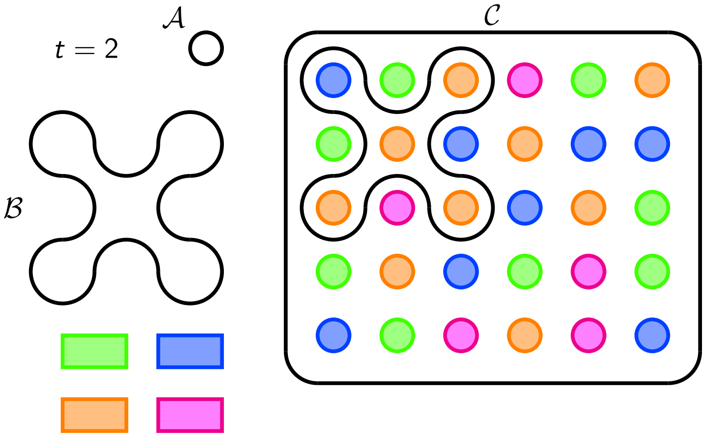

Structural Ramsey Theory

Examples.

1. [Fouché 1997/8] Finite simple graphs: $\tilde{t}(G) = \dfrac{n!}{|\mathrm{Aut}(G)|}$
2. [Fouché 1997/8] Finite posets: $\tilde{t}(P) = \dfrac{\mathrm{linext}(P)}{|\mathrm{Aut}(P)|}$
3. [KPT 2005] Finite equiv relations: $\tilde{t}(\rho) = \dfrac{\mathrm{convlo}(\rho)}{|\mathrm{Aut}(\rho)|}$

Categorification

$\mathbb{C}$ ... a category

$f \sim_A g$ ... $f = g \cdot \alpha$ for some $\alpha \in Aut(A)$

$\binom{B}{A}$ ... $= \hom(A, B)/{\sim_A}$

- $C \xrightarrow{\sim} (B)^A_{k,t}$: for every coloring $\chi : \binom{C}{A} \to k$ there is a $w \in \hom(B, C)$ such that $|\chi(w \cdot \binom{B}{A})| \leqslant t$.
- Small Ramsey degrees: $\tilde{t}(A) =$ minimal t such that for all k and B there is a C with $C \xrightarrow{\sim} (B)^A_{k,t}$; or ∞ if no such t exists;
- $\mathbb{C}$ has the Ramsey property: $\tilde{t}(A) = 1$ for every $A \in \mathrm{Ob}(\mathbb{C})$.

Ramsey degrees via essential partitions

$\mathbb{C}$ . . . locally small category whose morphisms are mono

A, B . . . objects of $\mathbb{C}$ such that $A \to B$

Definition. $\Lambda \in \mathrm{Part}\binom{B}{A}$ is *essential* if there is a $C \in \mathrm{Ob}(\mathbb{C})$ such that $B \to C$ and for every partition $\Pi \in \mathrm{Part}\binom{C}{A}$ there is a $w \in \hom(B, C)$ such that $\Lambda \succcurlyeq \ell_w^{-1}(\Pi)$.

$\mathrm{Ess}\binom{B}{A}$ . . . all essential partitions of $\binom{B}{A}$

Lemma. $\tilde{t}(A) = \sup_{B : A \to B} \min_{\Lambda \in \mathrm{Ess}\binom{B}{A}} |\Lambda|$.

Entropies on partitions

X ... a nonempty finite set

$\mathrm{Part}(X)$ ... lattice of partitions on X

$H_X : \mathrm{Part}(X) \to \mathbb{R} \cup \{\infty\}$

- $H_X(\Pi) \leqslant \log |\Pi|$;
- if $\Pi \cong \Lambda$ then $H_X(\Pi) = H_Y(\Lambda)$;
- $H_X(\Pi) = 0$ iff $\Pi = \{X\}$
- if $\Sigma \preccurlyeq \Pi$ then $H_X(\Sigma) \leqslant H_X(\Pi)$;
- $H_{X\times Y}(\Pi \times \Lambda) = H_X(\Pi) + H_Y(\Lambda)$.

Entropies on partitions

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Examples.

1. Boltzmann entropy:

$$H_X^{\mathrm{Bol}}(\Pi) = \log |\Pi|$$

2. Shannon entropy (where $p(\beta) = |\beta|/|X|$):

$$H_X^{\mathrm{Sha}}(\Pi) = -\sum_{\beta \in \Pi} p(\beta) \log p(\beta)$$

Ramsey entropy

$\mathbb{C}$ ... a small cat whose mor's are mono and homsets finite

H ... any entropy on partitions

Definition. Ramsey entropy based on H is $\tilde{r} : \mathrm{Ob}(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$ defined by:

$$\tilde{r}(X) = \inf_{A: X \to A} \sup_{B: A \to B} \min_{\Lambda \in \mathrm{Ess}\binom{B}{A}} H(\Lambda).$$

If $H = H^{\mathrm{Bol}}$ we refer to $\tilde{r}$ as the Ramsey-Boltzmann entropy

NB. $\tilde{r}$ can be defined for any small category whose morphisms are mono and homsets are finite!

Ramsey entropy

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H ... any entropy on partitions

Theorem. If $\mathbb{C}$ has finite Ramsey degrees then $\mathbb{C}$ admits a Ramsey entropy based on H (that is, $\tilde{r}(A) < \infty$ for all $A \in \mathrm{Ob}(\mathbb{C})$).

Ramsey entropy

$\mathbb{C}$ . . . a small cat whose mor's are mono and homsets finite

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Theorem.

- $\tilde{r}(A) \leqslant \log \tilde{t}(A)$
- $A \cong B \Rightarrow \tilde{r}(A) = \tilde{r}(B)$;
- if A is a subobject of a Ramsey object then $\tilde{r}(A) = 0$;
- $A \to B \Rightarrow \tilde{r}(A) \leqslant \tilde{r}(B)$;
- $\tilde{r}(A, B) \leqslant \tilde{r}(A) + \tilde{r}(B)$.

Ramsey-Boltzmann entropy

$\mathbb{C}$ ... a small cat whose mor's are mono and homsets finite

H^{Bol} ... the Boltzmann entropy on partitions

Theorem. Ramsey-Boltzman entropy is the maximal Ramsey entropy on a category.

Theorem. $\mathbb{C}$ admits a Ramsey-Boltzmann entropy (that is, $\tilde{r}(A) < \infty$ for all $A \in \mathrm{Ob}(\mathbb{C})$) if and only if $\mathbb{C}$ has finite Ramsey degrees.

Ramsey-Boltzmann entropy

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Theorem.

- $\tilde{r}(A) \leqslant \log \tilde{t}(A)$
- $A \cong B \Rightarrow \tilde{r}(A) = \tilde{r}(B)$;
- $\tilde{r}(A) = 0$ iff A is a subobject of a Ramsey object;
- $A \to B \Rightarrow \tilde{r}(A) \leqslant \tilde{r}(B)$;
- $\tilde{r}(A, B) = \tilde{r}(A) + \tilde{r}(B)$.

Concluding remarks

1. Can we use this point of view to detect the extremely elusive Ramsey property using strategies of information theory?

2. By duality, small dual Ramsey degrees also lead to the notion of entropy; compute Ramsey-Boltzmann entropy for the category of finite probability distributions and compare with Shannon entropy.