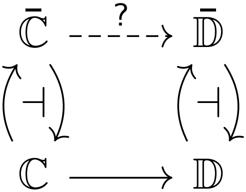
Lax Liftings & Lax Distributive Laws

Ezra Schoen

July 19, 2022

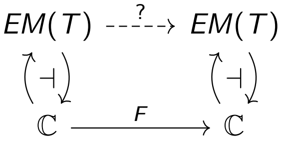
Liftings



Distributive Laws

- $$\begin{array}{ccc} EM(T) & \overset{?}{\dashrightarrow} & EM(T) \\ \uparrow\!\dashv\!\downarrow & & \uparrow\!\dashv\!\downarrow \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array}$$

Distributive Laws

- 

$\iff$ $FT \to TF$
$+$ conditions

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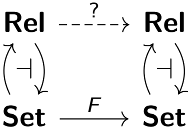
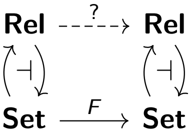
- $+$ many more

Relation Lifting



Relation Lifting



- At most one solution

Relation Lifting

$$\begin{array}{ccc} \mathbf{Rel} & \overset{?}{\dashrightarrow} & \mathbf{Rel} \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ \mathbf{Set} & \xrightarrow{F} & \mathbf{Set} \end{array}$$

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- Existence only if F preserves weak pullbacks

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Remedy

Make it lax!

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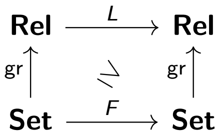

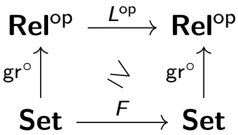

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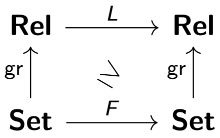

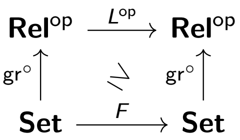

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Example

$FX = A \times X$

$$LR = \{(\langle a, x\rangle, \langle b, y\rangle) \mid xRy \text{ and } a \leq b\}$$

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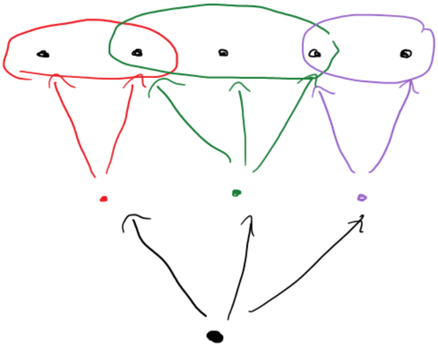
Theorem

Lax Liftings $\iff$ Lax Distributive Laws

Case study I: Neighborhoods

Definition

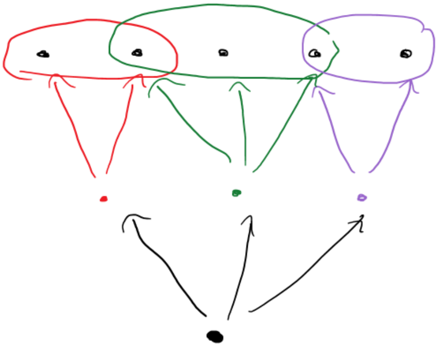
$$\mathcal{N}X = PPX$$



Definition

$$\mathcal{N}X = PPX$$

$$\mathcal{N}f : U \mapsto \{u \mid f^{-1}(u) \in U\}$$



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Liftings

A simple strategy:

1. Find some property of neighborhood systems
   - Being non-empty? ✗
   - Containing $\varnothing$? ✓
   - Containing X? ✓
2. Ask that it be preserved or reflected

Theorem

That's all you can do!

Case study II: Monotone Neighborhoods

Definition

$$\mathcal{M}X = \{U \in \mathcal{N}X \mid U \text{ upwards closed}\}$$

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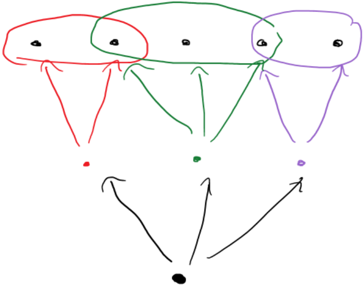
- Game logic
  - neighborhoods = Eloise's moves
  - points = Abelard's moves



The minimal lifting

$(U, V) \in \widetilde{\mathcal{M}}$ iff $\forall u \in U \exists v \in V : \forall y \in v \exists x \in u : xRy$ and vice versa

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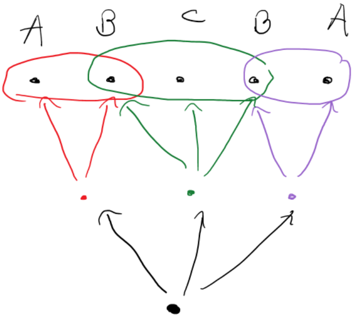
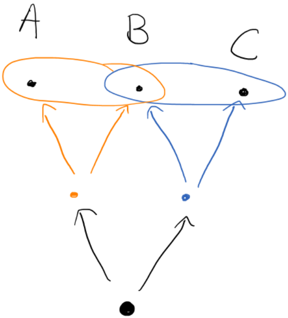
$\approx$ mutual reducibility of games.





Thank you