Simplicial Distributions and Quantum Contextuality

Cihan Okay

Bilkent University

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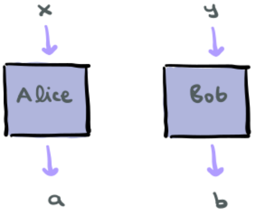
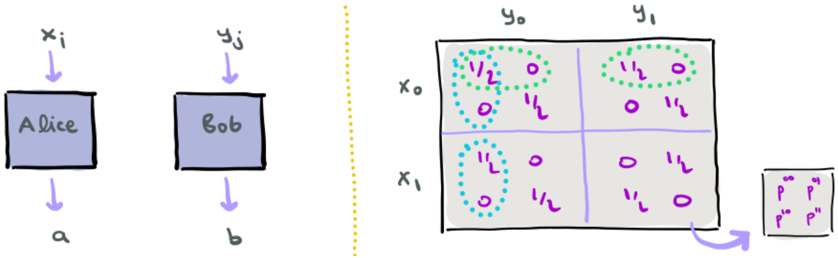
Based on arXiv:2204.06648 joint with Aziz Kharoof and Selman Ipek.

Can we represent measurement statistics in a topological way?





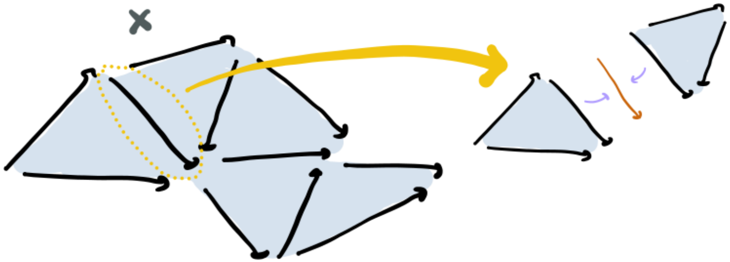
Outcome probabilities satisfy the **nonsignaling conditions**

$$p^{a}_{x_i} = \sum_b p^{ab}_{x_i y_0} = \sum_b p^{ab}_{x_i y_1} \qquad p^{b}_{y_j} = \sum_a p^{ab}_{x_0 y_j} = \sum_a p^{ab}_{x_1 y_j}$$

where $a, b \in \mathbb{Z}_2 = \{0, 1\}$.

sSet category of **simplicial sets**

- Objects: functors $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ where $[n] \mapsto X_n$.
- Morphisms: natural transformations $f : X \to Y$.



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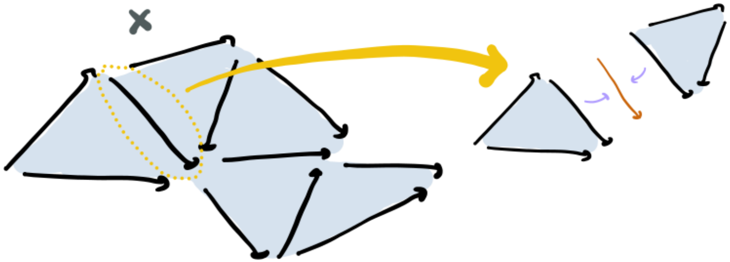
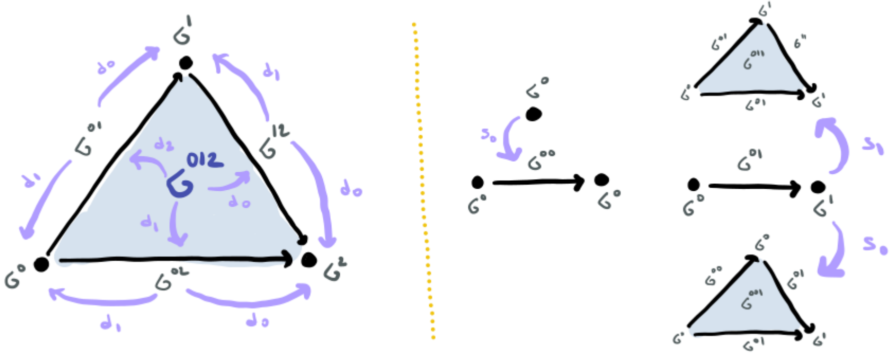


Simplex category Δ

- Objects: $[n] = \{0, 1, \cdots, n\}$ for $n \geq 0$.
- Morphisms: order preserving functions $\theta : [m] \to [n]$.

The ***n*-simplex** Δ^n

- Generating n-simplex: $\sigma^{01\cdots n}$.
- Face map d_i deletes the i-th index and degeneracy map s_j copies the j-th index.



The functor $D_R : \mathrm{Set} \to \mathrm{Set}$ for a semiring R (e.g. $\mathbb{R}_{\geq 0}$)

- $D_R(U)$ is the set of **R-distributions** on U, i.e. functions $p : U \to R$ of finite support such that

$$\sum_{u \in U} p(u) = 1.$$

- For $f : U \to V$ the function $D_R(f) : D_R(U) \to D_R(V)$ is defined by sending p to the distribution on V given by

$$v \mapsto \sum_{u \in f^{-1}(v)} p(u).$$

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Natural transformation $\delta : \mathrm{Id}_{\mathrm{Set}} \to D_R$ where $\delta_U : U \to D_R(U)$ sends $u \in U$ to the **deterministic** distribution

$$\delta^u(u') = \begin{cases} 1 & u' = u \\ 0 & \text{otherwise.} \end{cases}$$

A **simplicial scenario** consists of

- a space X of measurements where elements of X_n represent n-dimensional contexts,
- a space Y of outcomes where elements of Y_n represent n-dimensional outcomes.

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Space of distributions on Y

$$D_R(Y) : \Delta^{\mathrm{op}} \xrightarrow{Y} \mathrm{Set} \xrightarrow{D_R} \mathrm{Set}.$$

- Functor $D_R : \mathrm{sSet} \to \mathrm{sSet}$
- Natural transformation $\delta : \mathrm{Id}_{\mathrm{sSet}} \to D_R$

A **simplicial distribution** on a scenario (X, Y) is a map of simplicial sets

$$p : X \to D_R(Y).$$

Each n-context σ is assigned to a distribution p_σ on the set Y_n of n-outcomes such that

$$d_i p_\sigma = p_{d_i \sigma}$$

$$s_j p_\sigma = p_{s_j \sigma}.$$

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Canonical map

$$\Theta : D_R(\mathrm{sSet}(X, Y)) \to \mathrm{sSet}(X, D_R(Y))$$

induced by sending $r : X \to Y$ to the **deterministic** simplicial distribution

$$\delta^r : X \xrightarrow{r} Y \xrightarrow{\delta} D_R(Y).$$

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Otherwise it is called **noncontextual**.

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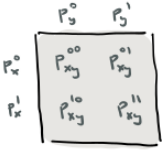
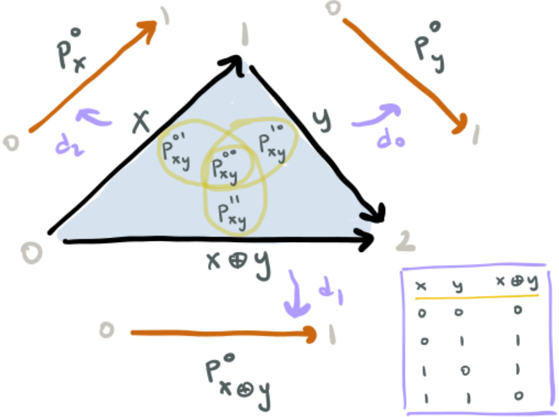
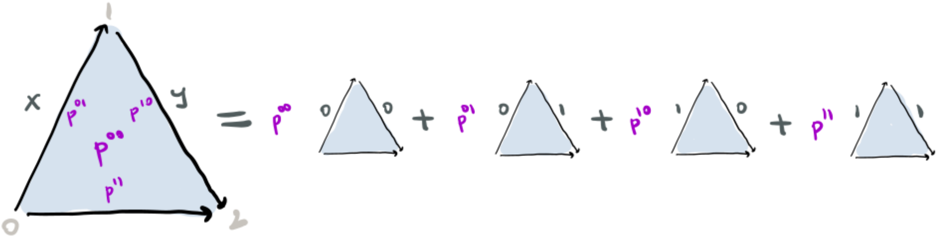
Generalizes "The sheaf-theoretic structure of non-locality and contextuality" by Abramsky and Brandenburger (2011).

Triangle scenario $p : \Delta^2 \to D_{\mathbb{R}_{\geq 0}}(N\mathbb{Z}_2)$



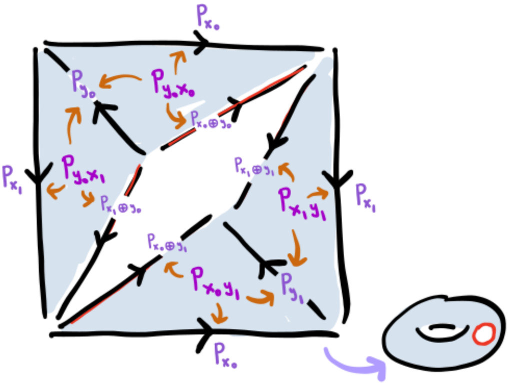


Every simplicial distribution on a triangle is noncontextual.



CHSH scenario $p : T^\circ \to D_{\mathbb{R}_{\geq 0}}(N\mathbb{Z}_2)$

	y_0		y_1	
x_0	1/2	0	1/2	0
	0	1/2	0	1/2
x_1	1/2	0	0	1/2
	0	1/2	1/2	0



Fine's theorem (1982) A distribution on the CHSH scenario is noncontextual if and only if it satisfies the CHSH inequalities

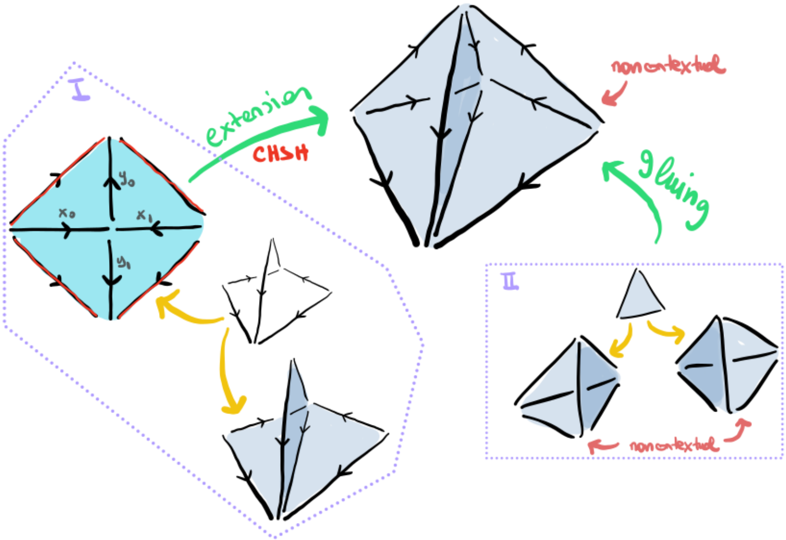
$$0 \leq p^0_{x_0 \oplus y_0} + p^0_{x_0 \oplus y_1} + p^0_{x_1 \oplus y_0} - p^0_{x_1 \oplus y_1} \leq 2$$

$$0 \leq p^0_{x_0 \oplus y_0} + p^0_{x_0 \oplus y_1} - p^0_{x_1 \oplus y_0} + p^0_{x_1 \oplus y_1} \leq 2$$

$$0 \leq p^0_{x_0 \oplus y_0} - p^0_{x_0 \oplus y_1} + p^0_{x_1 \oplus y_0} + p^0_{x_1 \oplus y_1} \leq 2$$

$$0 \leq -p^0_{x_0 \oplus y_0} + p^0_{x_0 \oplus y_1} + p^0_{x_1 \oplus y_0} + p^0_{x_1 \oplus y_1} \leq 2.$$

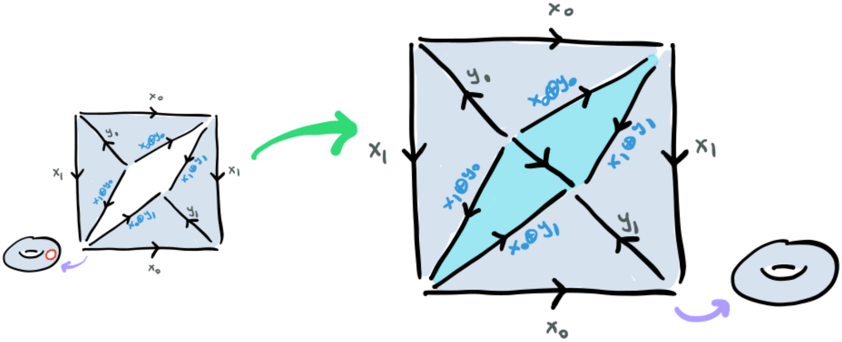
Proof[1]



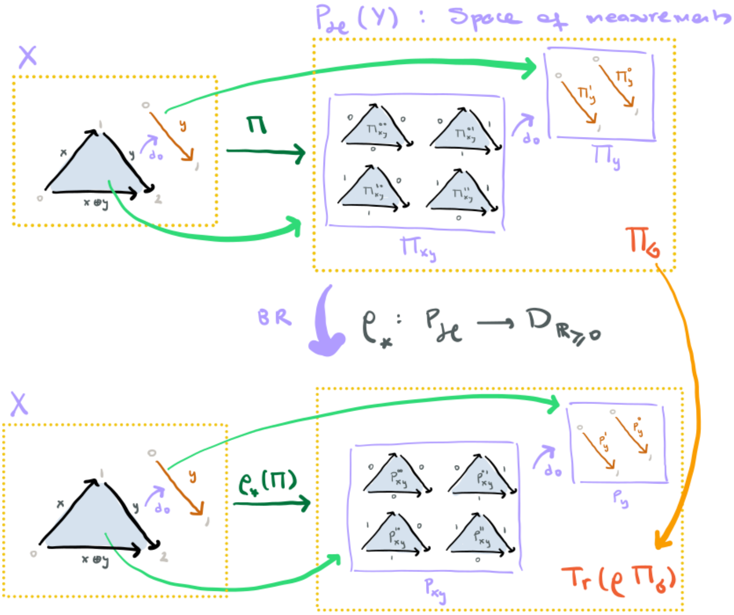
[1] Okay, Kharoof, Ipek 2022

A new characterization of contextuality[2]

A distribution on the CHSH scenario is noncontextual if and only if it extends to the torus.



[2] Okay, Kharoof, Ipek 2022



P_H(Y) : Space of measurements
X
Π
Π_xy
Π_y
Π_σ
do
x
y
x ⊕ y
BR
ρ_* : P_H → D_R≥0
X
ρ_*(Π)
P_xy
P_y
Tr(ρ Π_σ)

A quantum state ρ is **(non)contextual** with respect to $\Pi : X \to P_{\mathcal{H}}(Y)$ if the simplicial distribution $\rho_*(\Pi)$ is (non)contextual.

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Generalizes "Topological proofs of contextuality in quantum mechanics" by Okay, Roberts, Bartlett, and Raussendorf (2017).

Let $\mathcal{H}$ be a Hilbert space of dimension ≥ 3.

Gleason's theorem (1975) Every commutative diagram of simplicial sets

$$\begin{array}{ccc} P_{\mathcal{H}}(S^1) & \xrightarrow{f} & D_{\mathbb{R}_{\geq 0}}(S^1) \\ \uparrow{\scriptstyle \delta_{S^1}} & \nearrow{\scriptstyle \delta_{S^1}} & \\ S^1 & & \end{array}$$

is given by the simplicial Born rule, i.e $f = (\rho_*)_{S^1}$ for some quantum state ρ.

The **circle** S^1

- Generating 1-simplex: σ^{01}.
- Identifying relation: $d_0\sigma^{01} = d_1\sigma^{01}$.

Kochen–Specker theorem (1967) The following simplicial set map does not split

$$\delta_{S_1} : S^1 \to P_{\mathcal{H}}(S^1).$$

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Every quantum state ρ is contextual with respect to $\mathrm{Id}_{P_{\mathcal{H}}(S^1)} : P_{\mathcal{H}}(S^1) \to P_{\mathcal{H}}(S^1)$.

- $X = P_{\mathcal{H}}(S^1)$
- $Y = S^1$

Thank you for your attention!