**Applied Category Theory 2022**

$22^{\text{nd}}$ of July, University of Strathclyde, Glasgow

# Dialectica: fibrations and logical principles

*based on a joint work with* Davide Trotta (University of Pisa)
*&* Valeria de Paiva (Topos Institute)

*speaker* Matteo Spadetto (University of Leeds)

# Content

1. Notion of *Gödel doctrine* and motivation
2. Internal logic

# Gödel's Dialectica Interpretation

Dialectica Interpretation is based on a theory, called System $T$, in a many-sorted language $\mathcal{L}$ and such that any formula of $T$ is quantifier free. Whenever $A$ is a formula in the language of arithmetic, then we inductively define a formula $A^D$ in the language $\mathcal{L}$ of the form $\exists x.\forall y.A_D$, where $A_D$ is quantifier free. This interpretation satisfies the following:

**Theorem**

*If $HA$ proves a formula $A$, then $T$ proves $A_D(t, y)$ where $t$ is a sequence of closed terms.*

# Dialectica construction

De Paiva's notion of Dialectica category $\text{Dial}(\mathcal{C})$ associated to a category with finite limits $\mathcal{C}$ is the first attempt of internalising Gödel's Dialectica interpretation.

# Dialectica construction

De Paiva's notion of Dialectica category $\mathrm{Dial}(\mathcal{C})$ associated to a category with finite limits $\mathcal{C}$ is the first attempt of internalising Gödel's Dialectica interpretation.

An **object** of $\mathrm{Dial}(\mathcal{C})$ is a triple $(X, U, \alpha)$, which we think of as a formula $(\exists x)(\forall u)\alpha(x, u)$, where $\alpha$ is a subobject of $X \times U$ in $\mathcal{C}$.

# Dialectica construction

An **arrow** from $(\exists x)(\forall u)\alpha(x,u)$ to $(\exists y)(\forall v)\beta(y,v)$ is a pair $(F\colon X \longrightarrow Y,\ f\colon X \times V \longrightarrow U)$, i.e. a pair $(F(x) : Y,\ f(x,v) : U)$ of terms in context satisfying the condition:

$$\alpha(x, f(x,v)) \leq \beta(F(x), v)$$

between the reindexed subobjects, where the squares:

$$\begin{array}{ccc} \alpha(x, f(x,v)) & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[\langle \mathrm{pr}_X, f\rangle]{} & X \times U \end{array} \qquad \begin{array}{ccc} \beta(F(x), v) & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[F \times 1_V]{} & Y \times V \end{array}$$

are pullbacks.

# Dialectica construction

An **arrow** from $(\exists x)(\forall u)\alpha(x,u)$ to $(\exists y)(\forall v)\beta(y,v)$ is a pair $(F\colon X \longrightarrow Y,\ f\colon X \times V \longrightarrow U)$, i.e. a pair $(F(x) : Y,\ f(x,v) : U)$ of terms in context satisfying the condition:

$$\alpha(x, f(x,v)) \leq \beta(F(x), v)$$

between the reindexed subobjects, where the squares:

$$\begin{array}{ccc} \alpha(x,f(x,v)) & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[\langle \mathrm{pr}_X, f\rangle]{} & X \times U \end{array} \qquad \begin{array}{ccc} \beta(F(x),v) & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[F \times 1_V]{} & Y \times V \end{array}$$

are pullbacks.

The notion of morphism of $\mathrm{Dial}(\mathcal{C})$ is motivated by the definition of the dialectica interpretation for formulas of the form $A \to B$:

$$(A \to B)^D := (\exists F)(\exists f)(\forall x)(\forall v)(\ A_D(x, f(x,v)) \to B_D(F(x), v)\ ).$$

The action of $(-)^D$ on $A \to B$ is heuristically motivated by the Principle of Independence of Premise:

$$\top \vdash (\theta \to (\exists u)\eta(u)) \to (\exists u)(\theta \to \eta(u))$$

and Markov Principle:

$$\top \vdash \neg(\forall x)\phi(x) \to (\exists x)\neg\phi(x)$$

by which one can show that:

$$A^D \to B^D \dashv\vdash (A \to B)^D.$$

# (A presentation of) the generalised dialectica construction (proof-irrelevant setting)

Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a doctrine. The **dialectica doctrine** $\mathrm{Dial}(P)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ associated to $P$ is defined as follows:

- **Fibres.** The objects of $\mathrm{Dial}(P)(A)$ are 4-tuples $(A, X, U, \alpha)$ where $A, X$ and $U$ are objects of $\mathcal{C}$ and $\alpha \in P(A \times X \times U)$; it is the case that $(A, X, U, \alpha) \leq (A, Y, V, \beta)$ when there is a pair $(A \times X \xrightarrow{F} Y,\ A \times X \times V \xrightarrow{f} U)$ such that:

$$\alpha(a, x, f(a, x, v)) \vdash \beta(a, F(a, x), v).$$

- **Reindexing.** Whenever $g$ is an arrow $B \to A$ of $\mathcal{C}$, it is the case that $\mathrm{Dial}(P)(f)(A, X, U, \alpha)$ is:

$$(B, X, U, \alpha(g(b), x, u))$$

  of $\mathrm{Dial}(P)(B)$.

# Previous result

If $\mathcal{C}$ is cartesian closed, then a doctrine $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ is the dialectica completion of some doctrine $P''$ precisely when $P$ is a **Gödel doctrine**, that is (e.g.):

1. the doctrine $P$ is *existential* and *universal*;
2. the doctrine $P$ has enough *existential-free predicates*;
3. the existential-free objects of $P$ are stable under universal quantification, i.e. if $\alpha$ is an element of $P(A)$ and it is existential-free, then $\forall_{\mathrm{pr}}(\alpha)$ is existential-free for every projection pr from $A$;
4. the subdoctrine $P'\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ of the existential-free predicates of $P$ has enough *universal-free predicates*.

In this case, the doctrine $P''$ such that $\mathrm{Dial}(P'') \cong P$ is the full subdoctrine of the universal-free predicates of $P'$ (also called *quantifier-free predicates* of $P$).

# Existential doctrines and existential-free elements

A fibration $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ is **existential** if:

$$P_{\mathrm{pr}}\colon P(A) \to P(A \times B)$$

has a left adjoint $\exists_{\mathrm{pr}}\colon P(A \times B) \to P(A)$ for any projection $A \times B \xrightarrow{\mathrm{pr}} A$ of the base category (satisfying the BC condition).

# Existential doctrines and existential-free elements

A fibration $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ is **existential** if:

$$P_{\mathrm{pr}}\colon P(A) \to P(A \times B)$$

has a left adjoint $\exists_{\mathrm{pr}}\colon P(A \times B) \to P(A)$ for any projection $A \times B \xrightarrow{\mathrm{pr}} A$ of the base category (satisfying the BC condition).

Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ be an existential doctrine. We say that a predicate $\alpha(i)$ in $P(I)$ is **existential-free** if it enjoys the following universal property:

for every arrow $A \xrightarrow{f} I$ of $\mathcal{C}$ such that:

$$\alpha(f(a)) \vdash (\exists b\colon B)\beta(a,b)$$

in $P(A)$, where $\beta(a,b)$ is a predicate in $P(A \times B)$, there exist a unique arrow $A \xrightarrow{g} B$ such that:

$$\alpha(f(a)) \vdash \beta(a, g(a))$$

in $P(A)$.

# Logical principles

## Proposition (Prenex normal form)

*If a doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ *is a Gödel doctrine, then, for any predicate* $\alpha$ *in* $P(A)$*, it is the case that:*

$$\alpha(a) \dashv\vdash (\exists x\colon X)(\forall y\colon Y)\beta(x,y,a)$$

*where* $\beta$ *is a quantifier-free predicate in* $P(X \times Y \times A)$*.*

## Proposition (Skolemisation)

*If a doctrine* $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ *is a Gödel doctrine, then, for any predicate* $\beta$ *in* $P(X \times Y \times A)$*, it is the case that:*

$$(\forall x\colon X)(\exists y\colon Y)\beta(x,y,a) \dashv\vdash (\exists f\colon Y^X)(\forall x\colon X)\beta(x,\mathrm{ev}(f,x),a).$$

# Logical principles

## Theorem

*Let* $P\colon \mathcal{C}^{\text{op}} \to \mathbf{Pos}$ *be a Gödel doctrine. Then for every* $\psi_D$ *in* $P(I \times U \times X)$ *and* $\phi_D$ *in* $P(I \times V \times Y)$ *quantifier-free predicates of* $P$ *it is the case that:*

$$i : I \mid (\exists u)(\forall x)\psi_D(i, u, x) \vdash (\exists v)(\forall y)\phi_D(i, v, y)$$

*if and only if there exist* $I \times U \xrightarrow{f_0} V$ *and* $I \times U \times Y \xrightarrow{f_1} X$ *such that:*

$$i : I, u : U, y : Y \mid \psi_D(i, u, f_1(i, u, y)) \vdash \phi_D(i, f_0(i, u), y).$$

# Logical principles

## Theorem

*Let $P\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a Gödel doctrine. Then for every $\psi_D$ in $P(I \times U \times X)$ and $\phi_D$ in $P(I \times V \times Y)$ quantifier-free predicates of $P$ it is the case that:*

$$i : I \mid (\exists u)(\forall x)\psi_D(i,u,x) \vdash (\exists v)(\forall y)\phi_D(i,v,y)$$

*if and only if there exist $I \times U \xrightarrow{f_0} V$ and $I \times U \times Y \xrightarrow{f_1} X$ such that:*

$$i : I, u : U, y : Y \mid \psi_D(i,u,f_1(i,u,y)) \vdash \phi_D(i,f_0(i,u),y).$$

**Goal.** To say something about the internal logic of a Dialectica hyperdoctrine that relates it to the framework of the Dialectica translation.

# Dialectica rules

## Proposition

*Every Gödel hyperdoctrine* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{HeyAlg}$ *validates the* **Rule of Independence of Premise***:*

*if* $a : A \mid \top \vdash \alpha(a) \to (\exists b)\beta(a,b)$ *then* $a : A \mid \top \vdash (\exists b)(\alpha(a) \to \beta(a,b))$

*whenever* $\beta \in P(A \times B)$ *and* $\alpha \in P(A)$ *is an existential-free predicate.*

## Proposition

*Every Gödel hyperdoctrine* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{HeyAlg}$ *satisfies the following* **Markov Rule***:*

*if* $a : A \mid \top \vdash ((\forall b)\alpha(a,b)) \to \beta(a)$

*then* $a : A \mid \top \vdash (\exists b)(\alpha(a,b) \to \beta(a))$

*whenever* $\beta \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate.*

# Dialectica rules

The Rule of Independece of Premise and the Markov Rule are needed, in addition to the inference rules of intuitionistic first-order logic, in order to justify the definition of the Dialectica translation of formulas of arithmetic of the form $A \to B$. This fact underscores how faithful the modelling is.

# Dialectica principles

If in addition we assume that, for a Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{HeyAlg}$, the existential-free elements are closed under finite conjunction and implication, then it is the case that:

## Theorem

*The doctrine $P$ models the Principle of Independence of Premise:*

$$a : A \mid \top \vdash (\ \alpha(a) \to (\exists b)\beta(a,b)\ ) \to (\exists b)(\alpha(a) \to \beta(a,b))$$

*whenever $\beta \in P(A \times B)$ and $\alpha \in P(A)$ is an existential-free predicate; and the Markov Principle:*

$$a : A \mid \top \vdash (\ (\forall b)\alpha(a,b) \to \beta(a)\ ) \to (\exists b)(\alpha(a,b) \to \beta(a))$$

*whenever $\beta \in P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate.*

# Internal logic of a Dialectica doctrine

Any boolean doctrine satisfies the Principle of Independence of Premises and the Markov Principle, but in general these are not satisfied by a usual hyperdoctrine.
It turns out that *the set of deduction rules modelled by a Gödel hyperdoctrine is right in-between intuitionistic first-order and classical first-order logic*: it is powerful enough to guarantee the equivalences that justify the Dialectica interpretation of the implication.

# Internal logic of a Dialectica doctrine

Any boolean doctrine satisfies the Principle of Independence of Premises and the Markov Principle, but in general these are not satisfied by a usual hyperdoctrine.
It turns out that *the set of deduction rules modelled by a Gödel hyperdoctrine is right in-between intuitionistic first-order and classical first-order logic*: it is powerful enough to guarantee the equivalences that justify the Dialectica interpretation of the implication.

What about $A^D \to B^D \leftrightarrow (A \to B)^D$?

# Main result: soundness of $A^D \to B^D \leftrightarrow (A \to B)^D$

**Theorem**

*Let* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{HeyAlg}$ *be a Gödel hyperdoctrine such that:*

- *existential-free elements are closed with respect to implication and finite conjunction;*
- *falsehood* $\bot$ *is a quantifier-free predicate.*

*Then for every* $\psi_D$ *in* $P(I \times U \times X)$ *and* $\phi_D$ *in* $P(I \times V \times Y)$ *quantifier-free predicates of* $P$ *it is the case that the formula:*

$$i : I \mid \exists u.\forall x.\psi_D(i,u,x) \to \exists v.\forall y.\phi_D(i,v,y)$$

*is provably equivalent to:*

$$i : I \mid \exists f_0, f_1.\forall u, y.(\psi_D(i,u,f_1(i,u,y)) \to \phi_D(i,f_0(i,u),y)).$$

# References


- Trotta, Spadetto, de Paiva. *The Gödel fibration.* MFCS 2021. LIPIcs link here. arXiv 2104.14021 (extended version).
- Trotta, Spadetto, de Paiva. *Dialectica logical principles.* LFCS 2022. Springer link here. arXiv 2109.08064.
- Trotta, Spadetto, de Paiva. *Dialectica Principles via Gödel Doctrines.* arXiv 2205.07093.

# References


- F. Lawvere. 1969. *Adjointness in foundations.* Dialectica, 23:281–296.
- M. Hyland, P. Johnstone and A. Pitts. 1980. *Tripos theory.* Math. Proc. Camb. Phil. Soc., 88:205–232.
- V. de Paiva. 1991. *The Dialectica categories.* PhD Thesis, University of Cambridge.
- M. Hyland. 2002. *Proof theory in the abstract.* Annals of Pure and Applied Logic, 114:43–78.
- P. Hofstra. 2011. *The Dialectica monad and its cousins.* Models, logics, and higherdimensional categories: a tribute to the work of Mihály Makkai, 53:107-139
- D. Trotta and M. E. Maietti. 2021. *Generalised existential completions and their regular and exact completions.* Preprint.