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Extending monotones using Kan Extensions

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Resource Theories

This project demonstrates an application of Kan extensions to resource theories in extending monotones.

Resource theories model physical systems in which certain operations are considered to be 'free of cost' among of the set of all operations.

Placing a glass of chilled water at room temperature warms up the water to the ambient temperature. Such transformations are considered to be free.

To prepare a glass of chilled water, one requires non-free transformations, such as a fridge, which consumes electricity.

Randomness as a resource

Randomness is a computational resource of practical use.

Randomness is also used in computer algorithms to solve certain problems.

Cryptographic protocols use randomness as an essential resource for establishing secure communication of devices by generating random keys.

For such systems which utilizes randomness, the transformations which does not increase randomness are considered to be free.

Monotones

A central question in the resource-theoretic modelling of systems is: given two resources, is there a free transformation to convert one resource into the other?

Answering this question imposes a preorder on resources which captures their value or usefulness.

A monotone for a resource theory is an order-preserving map from the set of all the resources into $[0, \infty]$.

A monotone assigns a value to each resource such that the assignment is compatible with the preorder, viz. with the usefulness of resources.

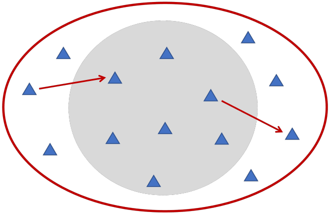
Entropy is used as measure of randomness: in particular, Shannon entropy quantifies randomness in that it expresses the average surprisal on the outcome of a random experiment.

Extending monotones

Consider a resource theory which embeds in a larger theory...

– for example classical theories embedding in quantum theories, pure state quantum theories embedding in mixed states theories.

Can a monotone M for the smaller theory be used to quantify the resources in the larger resource theory?



It is possible to extend the monotone M to give optimal upper and lower bounds respectively on the value of all the resources in the larger theory.

Framework for extending monotones

Gour and Tomamichel[1] provided a set-theoretical framework for extending monotones from a subset of resources to the entire set of resources.

They provided the forumale for computing extensions and proved the properties of the extensions thus computed.

Similar construction also introduced by Gonda and Spekkens[2].

Our observation: The monotone extensions are special cases of more general categorical concepts, called (point-wise) left and right Kan extensions

We first define the properties of the monotone extensions using Kan extensions. The properties in turn determine the formulae to compute the extensions.

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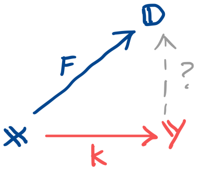
[1] Gour, Tomamichel (2020) "Optimal extensions of resource measures and their applications"

[2] Gonda and Spekkens (2019) "Monotones in general resource theories"

Right Kan extension



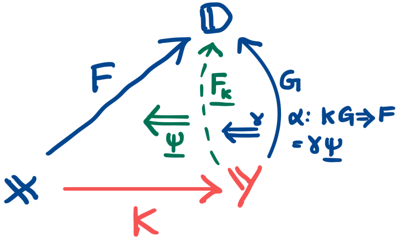
Let $F : \mathbb{X} \to \mathbb{D}$ be a functor.

Suppose we want to construct a functor $\mathbb{Y} \to \mathbb{D}$ based the information carried by $F : \mathbb{X} \to \mathbb{D}$

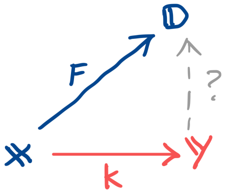
Given a functor from $K : \mathbb{X} \to \mathbb{Y}$, we can ask 'what' is (information theoretically) the most conservative such functor that can be constructed from $\mathbb{Y} \to \mathbb{D}$?

Right Kan extension (cont..)

Right Kan extension of F along K is a functor $\underline{F}_K : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\underline{\psi} : K\underline{F}_K \Rightarrow F$ which is universal.



Left Kan extension



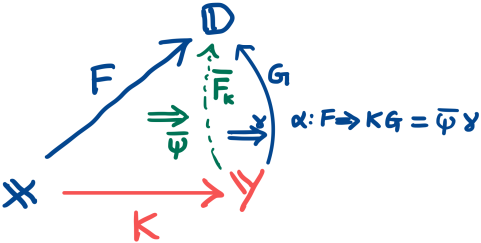
Let $F : \mathbb{X} \to \mathbb{D}$ be a functor.

Suppose we want to construct a functor $\mathbb{Y} \to \mathbb{D}$ based the information carried by $F : \mathbb{X} \to \mathbb{D}$

Given a functor from $K : \mathbb{X} \to \mathbb{Y}$, we can ask 'what' is (information theoretically) the most liberal such functor that can be constructed from $\mathbb{Y} \to \mathbb{D}$?

Left Kan extension

Left Kan extension of F along K is a functor $\overline{F}_K : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\underline{\psi} : F \Rightarrow K\overline{F}_K$ which is couniversal.

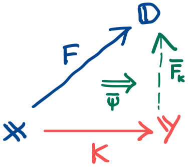


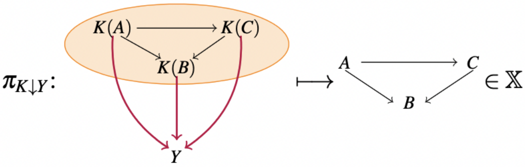
Computing left Kan extension

Given functors $F : \mathbb{X} \to \mathbb{D}$ and $K : \mathbb{X} \to \mathbb{Y}$, if the category $\mathbb{D}$ is cocomplete, then the left Kan extension $\overline{F}_K$ exists and is:-

$$\forall Y \in \mathbb{Y}, \quad \overline{F}_K(Y) := \mathrm{colim}(K \downarrow Y \xrightarrow{\pi_{K\downarrow Y}} \mathbb{X} \xrightarrow{F} \mathbb{D}) \tag{1}$$

with the natural transformation $\overline{\psi}$ extracted from colimiting cocones in $\mathbb{D}$.



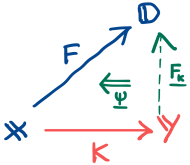


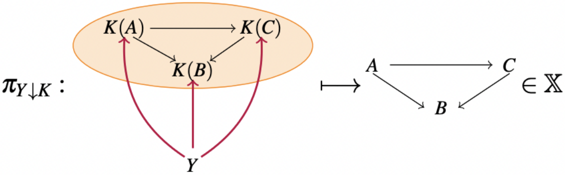
Computing right Kan extension

Given functors $F : \mathbb{X} \to \mathbb{D}$ and $K : \mathbb{X} \to \mathbb{Y}$, if the category $\mathbb{D}$ is complete, then the right Kan extension $\underline{F}_K$ exists and is:-

$$\forall Y \in \mathbb{Y}, \quad \underline{F}_K(Y) := \lim(Y \downarrow K \xrightarrow{\pi_{Y \downarrow K}} \mathbb{X} \xrightarrow{F} \mathbb{D}) \tag{2}$$

with the natural transformation $\underline{\psi}$ extracted from limiting cones in $\mathbb{D}$.





Categorical Framework for Monontone extensions

(a) Right Kan Extension (b) Left Kan Extension

To set up monotone extensions as Kan extensions of a monotone, we need:

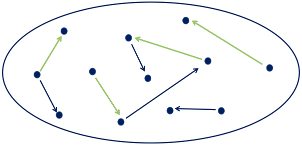
(a) a categorical description of resource theories,

(b) functors for resource theories, and

(c) monotones described as such a functor.

Resource Theories as Partitioned Categories (pCats)

Definition: A **partitioned category (pCat)** $(\mathbb{X}, \mathbb{X}_f)$ consists of a category $\mathbb{X}$ and a chosen subcategory $\mathbb{X}_f$ of free transformations with the inclusion being bijective on objects.



Objects of $\mathbb{X}$ are interpreted as **resources**.

Maps of $\mathbb{X}$ are interpreted as **resource transformations**.

$\mathbb{X}_f$ is interpreted as the category of free transformations.

The notion of partitioned categories is based on partitioned symmetric monoidal categories[3]

[3] Coecke, Fritz, Spekkens (2016) "A mathematical theory of resources"

Example: Resource theory of randomness

(Rand, Detmn):

Resources: (X, p) where X is a finite set and p is a probability distribution over X.

Resource Transformations: $M : (X, p) \to (Y, q)$ is a real $|X| \times |Y|$ row stochastic matrix (rows sum to 1) such that $pM = q$.

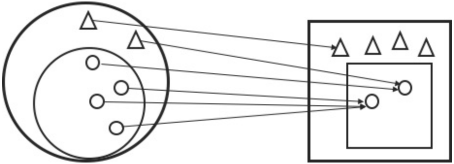
Identity transformations: Identity matrices

Composition: Suppose $(X, p) \xrightarrow{M} (Y, q) \xrightarrow{N} (Z, s)$, then $(X, p) \xrightarrow{MN} (Z, s)$ is defined as the matrix multiplication

Free transformations: A resource transformation $(X, p) \xrightarrow{M} (Y, q)$ is free if it is deterministic, that is, M is simply a function $X \to Y$.

pCat functors

A functor of partitioned categories (pCat functor) $F : (\mathbb{X}, \mathbb{X}_\mathsf{f}) \to (\mathbb{Y}, \mathbb{Y}_\mathsf{f})$ is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that F preserves free transformations i.e., if $f \in \mathbb{X}_\mathsf{f}$ then $F(f) \in \mathbb{Y}_\mathsf{f}$.



A pCat functor $F : (\mathbb{X}, \mathbb{X}_\mathsf{f}) \to (\mathbb{Y}, \mathbb{Y}_\mathsf{f})$ is full if $F : \mathbb{X} \to \mathbb{Y}$ is full, and F is faithful if $F : \mathbb{X} \to \mathbb{Y}$ is faithful.

Preorder collapse

Free transformations of a resource theory define a preorder on the resources.

A preorder $(X, \leq)$ can be encoded as the pCat $(\mathsf{Chaos}_X, \leq_X)$:
- Chaos_X is the indiscrete category with the elements of X as its objects
- The transformation $A \to B \in \mathsf{order}_{\mathbb{X}}$ if "$A \ \leq \ B$" is true

This means that the information about the existence of a free transformation between two resources can be encoded as a pCat functor:

Given a resource theory $(\mathbb{X}, \mathbb{X}_\mathsf{f})$ and a preorder $(\mathsf{ob}(\mathbb{X}), \mathsf{order})$, a preorder collapse of the resource theory $(\mathbb{X}, \mathbb{X}_\mathsf{f})$ is the pCat functor:

$$(\mathbb{X}, \mathbb{X}_\mathsf{f}) \to (\mathsf{chaos}_{\mathbb{X}}, \mathsf{order}_{\mathbb{X}})$$

A monotone is a preorder collapse

A monotone for a resource theory $(\mathbb{X}, \mathbb{X}_{\mathsf{f}})$ is a preorder collapse into $([0, \infty], \leq)$

$$M : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to \left(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]}\right)$$

If $f : A \to B \in \mathbb{X}_{\mathsf{f}}$, then $M(A) \leq M(B)$

An op-monotone is a contravariant monotone, that is,

$$F : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to \left(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]}\right)^{\mathsf{op}} := \left(\mathsf{chaos}^{\mathsf{op}}_{[0,\infty]}, \geq_{[0,\infty]}\right)$$

If $f : A \to B \in \mathbb{X}_{\mathsf{f}}$, then $M(A) \geq M(B)$

Example: Shannon Entropy

Shannon entropy defines a monotone for the resource theory of randomness, $(\mathsf{Rand}, \mathsf{Detmn})$:

$\mathsf{Shannon} : \mathsf{Rand} \to \mathsf{chaos}_{[0,\infty]}$ as follows:

- For all finite probability distributions, $(X, p) \in \mathsf{Rand}$, $\mathsf{Shannon}(p) := H(p)$ where $H(p)$ is the Shannon entropy of p:

$$H(p) := -\sum_{1 \leq i \leq |X|} p_i \log p_i$$

- For all $(X, p) \xrightarrow{\mathsf{M}} (X, q) \in \mathsf{Rand}$, then $F(M)$ is the unique arrow $H(p) \to H(q)$

Assembling the pieces together

Now that we have set up monotones as pCat functors, it is straightforward to describe a monotone extension as Kan extension of the monotone along another pCat functor.

Observe that a monotone M is concerned only with the free transformations: if $f : A \to B$ is free then, $M(A) \leq M(B)$.

In order to extend a monotone M it suffices to extend $M_{\mathsf{f}} : \mathbb{X}_{\mathsf{f}} \to \leq_{[0,\infty]}$ which is defined as follows :-

$$M_{\mathsf{f}} : \mathbb{X}_{\mathsf{f}} \to \leq_{[0,\infty]}; \quad A \xrightarrow{f} B \mapsto M(A) \leq M(B)$$

Extending Monotones

Let $M : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to (\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$. Let $(\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \xrightarrow{K} (\mathbb{Y}, \mathbb{Y}_{\mathsf{f}})$ be a pCat functor.

The minimal extension of M[4] along K is the right Kan extension of the functor $M_{\mathsf{f}} : \mathbb{X}_{\mathsf{f}} \to \leq_{[0,\infty]}$ along the functor $K_{\mathsf{f}} : \mathbb{X}_{\mathsf{f}} \to \mathbb{Y}_{\mathsf{f}}$

The maximal extension of M along K is the left Kan extension of the functor $M_{\mathsf{f}} : \mathbb{X}_f \to \leq_{[0,\infty]}$ along the functor $K_{\mathsf{f}} : \mathbb{X}_{\mathsf{f}} \to \mathbb{Y}_{\mathsf{f}}$.

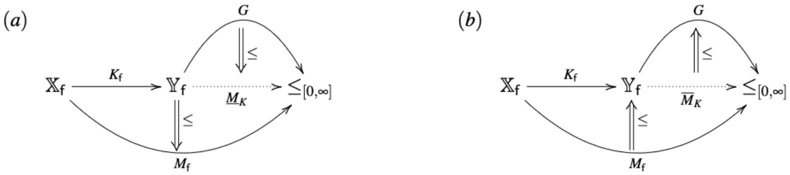


Figure 6: (a) Minimal (right Kan) extension (b) Maximal (left Kan) extension

[4]We follow the naming convention by Gour and Tomamichel

Properties of monotone extensions

Let $\underline{M}_K$ and $\overline{M}_K$ be minimal and maximal extensions of a monotone M for $(\mathbb{X}, \mathbb{X}_\mathsf{f})$ along a pCat functor $K : \mathbb{X} \to \mathbb{Y}$.

(a) Reduction: For all $X \in \mathbb{X}$,

$$\underline{M}_K(K_\mathsf{f}(X)) \leq M(X) \leq \overline{M}_K(K_\mathsf{f}(X))$$

(b) Monotonicity: For all $f : A \to B \in \mathbb{Y}_\mathsf{f}$,

$$\underline{M}_K(A) \leq \underline{M}_K(B) \text{ and } \overline{M}_K(A) \leq \overline{M}_K(B)$$

(c) Optimality: $\underline{M}_K$ and $\overline{M}_K$ are optimal.

– for any other monotone G for $(\mathbb{Y}, \mathbb{Y}_\mathsf{f})$ such that for all $X \in \mathbb{X}$, $G(K(X)) \leq M(X)$, we have that $G(Y) \leq \underline{M}_K(Y)$

– for any other monotone G' for $(\mathbb{Y}, \mathbb{Y}_\mathsf{f})$ such that for all $X \in \mathbb{X}$, $M(X) \leq G'(K(X))$, we have that $\overline{M}_K(Y) \leq G'(Y)$

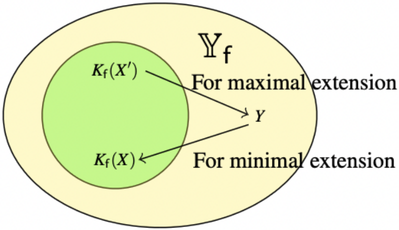
Computing monotone extensions

For all $Y \in \mathbb{Y}$, the minimal extension $\underline{M}_K(Y) : \mathbb{Y}_{\mathsf{f}} \to \leq_{[0,\infty]}$ is:

$$\underline{M}_K(Y) := \lim(\pi_{Y\downarrow K} M_{\mathsf{f}}) = \inf\{M(X) \mid Y \to K(X) \in \mathbb{Y}_{\mathsf{f}}\}$$

For all $Y \in \mathbb{Y}$, the maximal extension $\overline{M}_K(Y) : \mathbb{Y}_{\mathsf{f}} \to \leq_{[0,\infty]}$ is:

$$\overline{M}_K(Y) := \operatorname{colim}(\pi_{K\downarrow Y} M_{\mathsf{f}}) = \sup\{M(X) \mid K(X) \to Y \in \mathbb{Y}_{\mathsf{f}}\}$$



Examples

Extending the entanglement monotone given by Schmidt number from pure bipartite states to mixed states

Extending Shannon Entropy from classical states to states of a general physical theory

Extending classical divergences to quantum setting

All the above extensions are along a full and faithful inclusion functor.

More examples to be explored...

Summary

In this project, we generalized the set-theoretic framework for monotone extensions using Kan extensions.

Our generalized framework can be used to extend a monotone along an arbitrary functor instead of just a full and faithful embedding.

Moreover, our framework generalizes extensions to arbitrary preorders with suprema and infima.

Cockett, Geng, Scandolo, Srinivasan (2022) "Extending resource monotones using Kan extensions" arXiv:2206.09784