Enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities

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Canada

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- In [6, 7], Lawvere and Linton established a structure–semantics adjunction between Lawvere theories and *tractable* **Set**-valued functors. For a complete symmetric monoidal closed category $\mathscr{V}$, Dubuc [4] established a structure–semantics adjunction between $\mathscr{V}$*-theories* and *tractable* $\mathscr{V}$-valued $\mathscr{V}$-functors.

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- Building on work of Power and Nishizawa [11], Bourke and Garner [2] recently showed that if $\mathscr{J} \hookrightarrow \mathscr{C}$ is any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ enriched over a locally presentable closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$*-theories* and $\mathscr{J}$*-nervous* $\mathscr{V}$-monads on $\mathscr{C}$.

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- Yes! Given a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ enriched over a closed category $\mathscr{V}$, we will identify hypotheses on these data that entail a structure–semantics adjunction, a monad–theory equivalence, and a rich theory of *presentations* for enriched monads and theories.

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- Moral of the story: these "nice" subcategories of arities admit extremely rich and useful treatments of enriched algebra, in some completely new settings.

Basic definitions

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- We fix a **subcategory of arities** $j : \mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a small, full, and dense sub-$\mathscr{V}$-category, in a $\mathscr{V}$-category $\mathscr{C}$ enriched over a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$.

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- We have a fully faithful $\mathscr{V}$-functor

$$N_j : \mathscr{C} \to [\mathscr{J}^{\mathsf{op}}, \mathscr{V}]$$

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- Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. The $\mathscr{V}$-category $\mathscr{T}$-Alg of $\mathscr{T}$-**algebras** is defined by the following pullback in $\mathscr{V}$-CAT:

$$\begin{array}{ccc} \mathscr{T}\text{-Alg} & \longrightarrow & [\mathscr{T}, \mathscr{V}] \\ {\scriptstyle U^{\mathscr{T}}}\big\downarrow & & \big\downarrow{\scriptstyle [\tau,1]} \\ \mathscr{C} & \xrightarrow[N_j]{} & [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]. \end{array}$$

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- A $\mathscr{J}$-pretheory $\mathscr{T}$ is **admissible** if $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ has a left adjoint (i.e. if $\mathscr{T}$ **admits** free algebras).

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- The subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ is **amenable** if every $\mathscr{J}$-theory is admissible, and is **strongly amenable** if every $\mathscr{J}$-pretheory $\mathscr{T}$ is admissible.

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- An object $G : \mathscr{A} \to \mathscr{C}$ of $\mathscr{V}$-CAT$/\mathscr{C}$ is a **$\mathscr{J}$-tractable $\mathscr{V}$-category over** $\mathscr{C}$ if $\mathscr{C}$ admits the weighted limit $\{\mathscr{C}(J, G-), G\}$ for each $J \in \mathbf{ob}\,\mathscr{J}$.

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- Let $\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ be the full subcategory of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ consisting of the *admissible* $\mathscr{J}$-pretheories. We can then define a **semantics** functor

$$\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

by

$$\mathbf{Sem}\,\mathscr{T} = \left(U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}\right).$$

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- Let $G : \mathscr{A} \to \mathscr{C}$ be a $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$. We define a $\mathscr{J}$-theory $\tau_G : \mathscr{J}^{\mathbf{op}} \to \mathbf{Str}\,G$, the **$\mathscr{J}$-structure of G**, by taking the (identity-on-objects, fully faithful) factorization of the composite $\mathscr{V}$-functor

$$\begin{array}{ccccc} \mathscr{J}^{\mathbf{op}} & \xrightarrow{j^{\mathbf{op}}} & \mathscr{C}^{\mathbf{op}} & \xrightarrow{N_{G^{\mathbf{op}}}} & [\mathscr{A}, \mathscr{V}]. \\ & {}_{\tau_G}\searrow & & \nearrow & \\ & & \mathbf{Str}\,G & & \end{array}$$

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Theorem

Let $\mathscr{J} \hookrightarrow \mathscr{C}$ *be an amenable subcategory of arities. Then the semantics functor* $\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ *has a left adjoint* $\mathbf{Str}$ *that sends each* $\mathscr{J}$*-tractable* $\mathscr{V}$*-category over* $\mathscr{C}$ *to its* $\mathscr{J}$*-structure.*

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A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is $\mathscr{J}$**-nervous** if $\mathbb{T} \cong \Psi\mathscr{T}$ for some admissible $\mathscr{J}$-pretheory $\mathscr{T}$ (there are other equivalent definitions too).

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Theorem

Let $\mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. Then the idempotent monad–pretheory adjunction $\Psi \dashv \Phi$ restricts to an adjoint equivalence

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$$

between $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads, which commutes with semantics in an appropriate sense.

Additional consequences of strong amenability

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Theorem

Let $\mathscr{C}$ be complete and cocomplete, and suppose that $\mathscr{J} \hookrightarrow \mathscr{C}$ is strongly amenable. Then:

1. $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$, $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$, *and* $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *are all* **algebraically** *cocomplete.*
2. $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *is monadic over a category of* $\mathscr{J}$**-signatures**, *so that* $\mathscr{J}$*-nervous* $\mathscr{V}$*-monads admit a rich and useful theory of* **presentations**.

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  - The "strongly finitary" subcategory of arities $\mathbf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the terminal object (i.e. the natural number arities) in any complete and cocomplete cartesian closed category $\mathscr{V}$.

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  - The Yoneda embedding $\mathbf{y} : \mathscr{A}^{\mathbf{op}} \hookrightarrow [\mathscr{A}, \mathscr{V}]$ for a small $\mathscr{V}$-category $\mathscr{A}$.

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For suitable choices of eleutheric (and bounded) $\mathscr{J} \hookrightarrow \mathscr{C}$, we then recover the enriched structure–semantics adjunctions and monad–theory equivalences of Lawvere and Linton [6, 7], Dubuc [4], Lucyshyn-Wright [8], and Bourke–Garner [2].

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By omitting eleuthericity and strengthening the notion of boundedness, we can also obtain other classes of examples of strongly amenable subcategories of arities.

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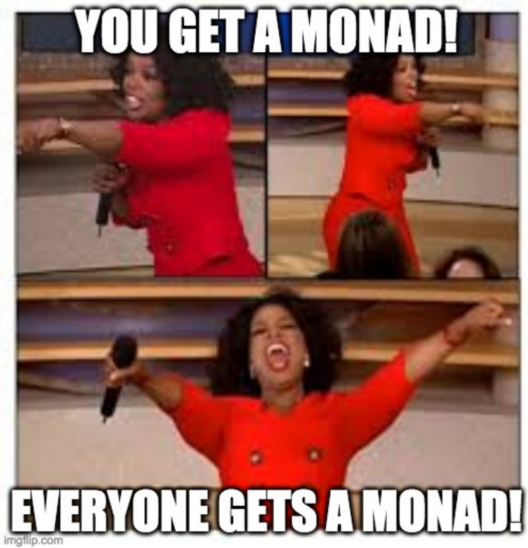
All of these closed categories will now admit extremely rich and useful treatments of enriched algebra. For example, we can now construct enriched monads and theories almost "at will" on many closed categories of relevance for topology, differential geometry, analysis, and programming language semantics, many of which we have seen at ACT (**sSet**, **Poly**, **Qbs**, **DCPO**...).


YOU GET A MONAD!
EVERYONE GETS A MONAD!
imgflip.com

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- As seen in Rory's talk, we also have extremely flexible and practical methods for easily constructing enriched monads and theories in these classes of examples, using enriched operations and equations.
- If you have a favourite symmetric monoidal closed category $\mathscr{V}$ (or $\mathscr{V}$-category) on which you want to construct and study enriched monads, please talk to us! :)

Thank you!

Comments and questions are welcome!

References I

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