Unification of Modal Logic via Topological Categories

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Outline

Introduction

Unification of Semantics

Reasoning Patterns as Universal Structures

Introduction

Brief Intro to Modal Logic

One way of looking at modal logic:

Describes inference of knowledge based on information structures.

- Syntax: *Uniform*
  - Add *modalities* to the language of propositional logic: $\Box\varphi$.
- Semantics: *Polymorphic*
  - Relations, topology, neighbourhoods, algebraic, etc.

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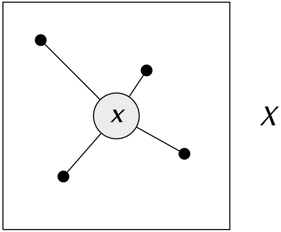
Brief Intro to Modal Logic

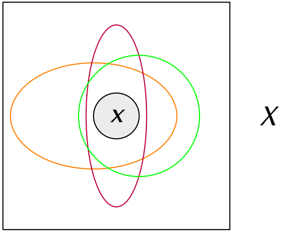
The most broadly used way is to assign a *relation* to an agent:



Formally, we have

$$X, R, x \vDash \Box\varphi \iff \forall y \in R[x]. X, R, y \vDash \varphi, \quad \text{or} \quad [\![\Box\varphi]\!] = R^{+}[\![\varphi]\!].$$

An increasingly explored way is to assign a *topology* to an agent:



Formally, we have

$$X, \tau, x \vDash \Box\varphi \iff \exists U \in \tau[x]. U \subseteq [\![\varphi]\!] \quad \text{or} \quad [\![\Box\varphi]\!] = \iota_\tau [\![\varphi]\!].$$

Brief Intro to Modal Logic

Many more reasoning patterns are developed (at different contexts):

- Multi-agency: $\Box_a \Box_b \varphi$;
- Modal dependence: $K_a b \wedge \Box_b \varphi \rightarrow \Box_a \varphi$;
- Different types of group agency $\Box_G \varphi$;
- Logical dynamics $[!\varphi]\psi$, $[\mathsf{E}, i]\varphi$.

Problems at Hand

Can we ...

- resolve the discrepancy: uniform syntax & polyform semantics?
- systematically describe the connections of these semantics?
- generalise the above reasoning patterns to all semantics?
- develop structural results for modal logic?

Our work: Provide a *single* theoretic framework, based on *topological categories* to address them all.

Cornerstone: All the mentioned classes of semantics are instances of topological categories.

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Unification of Semantics

Topological Categories

Usually, topological categories are described as certain *fibrations* or *opfibrations* satisfying certain lifting properties,

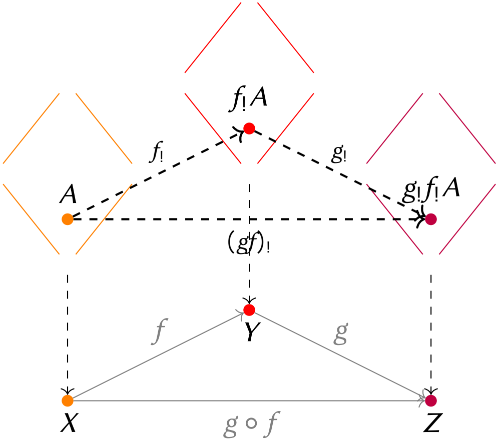
$$|-| : \mathcal{A} \to \mathbf{Set}.$$

Theorem

A (fibre-small) topological category is equivalent to a functor

$$\mathcal{A}_{-} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{InfL}, \quad \text{or} \quad \mathcal{A}_{-} : \mathbf{Set} \to \mathbf{SupL}.$$

The generic picture is as follows:



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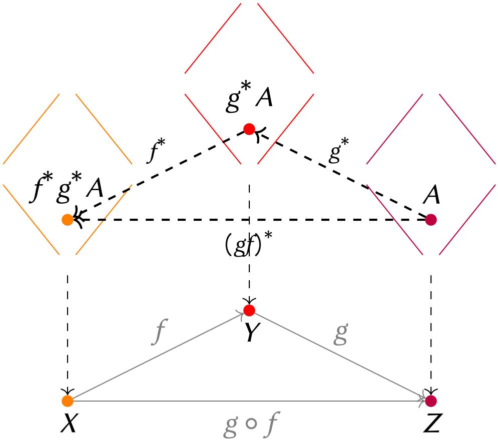


Semantic Categories Are Topological

Main examples of categories of semantics are topological:

- **Kr**, binary relations with monotone funtions. Fibre over X is $\wp(X \times X)$, with pullback along $f : X \to Y$ given by

$$(x, x') \in f^* R \Leftrightarrow (fx, fx') \in R.$$

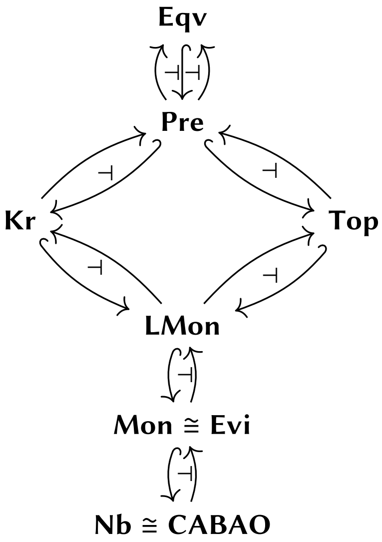
- **Top**. Fibres over X are topologies with *reverse* inclusion, with pullback along $f : X \to Y$ given by

$$U \in f^* \tau \Leftrightarrow \exists V \in \tau . U = f^{-1}(V).$$

- Others have similar structures.

A Larger Picture

Functors between topological categories as *model transformations*:



Reasoning Patterns as Universal Structures

Correspondence between Syntax and Semantics

Different modal reasoning patterns now corresponds to universal structures within topological categories:

Modal Dependence	↔	Partial Orders within Fibres
Group Structure	↔	Lattice Operations within fibres
Logical Dynamics	↔	Connections between fibres

Theorem

Model transformation F preserves the interpretation of reasoning pattern A ⟺ *F commutes with the semantic structure A corresponds to.*

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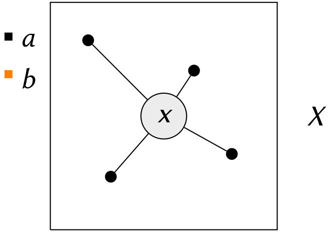
Modal Dependence

The order in fibres signifies the *epistemic strength* of each agent:



This signifies *dependence* in two agents' knowledge,

$$x \vDash \Box_a \varphi \to \Box_b \varphi.$$

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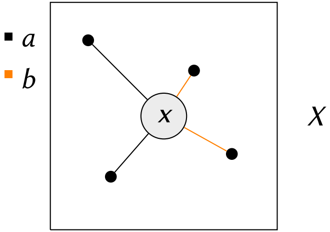

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Group Structure

Group structure is about ways of combining agents' information.
There are two canonical ways to do this, $\bigwedge, \bigvee : \mathcal{A}_X^G \to \mathcal{A}_X$:

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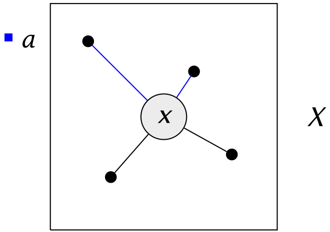

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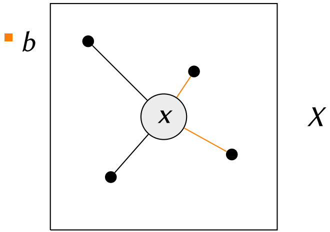

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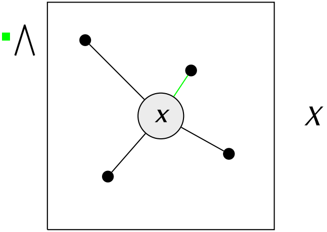

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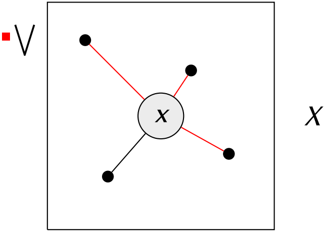

$\bigvee_{a \in G} A_a$ collects agents' *uncertainties*, and is the universal agent that if it knows, then each individual in the group knows.

Logical dynamics is to reason when *new information coming in.*

The most simple type of dynamics is *public announcement.*

Announcing φ, or $!\varphi$, restricts the domain:

Mathematically, we utilise the *fibre connections* to generate the interpretation of dynamic operators: $i : S \hookrightarrow X$

$$[\![[!\varphi]\psi]\!]_A := \forall_i [\![\psi]\!]_{i^*A}, \quad [\![\langle !\varphi \rangle \psi]\!]_A := \exists_i [\![\psi]\!]_{i^*A}.$$

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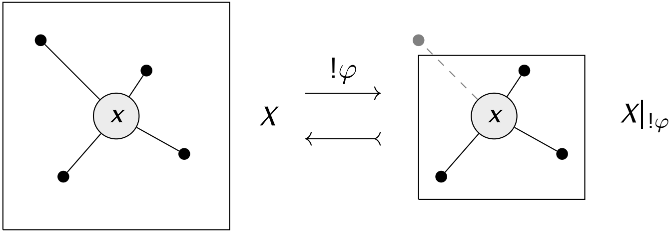

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Main contributions:

(I) Describing topological categories as *indexed categories*.

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(III) Obtained structural results for modal logic.

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Thanks for Listening!

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