An Example of Lawvere's Entropy Framework

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Lawvere's Framework

Lawvere postulates a category $\mathcal{P}$ wherein:

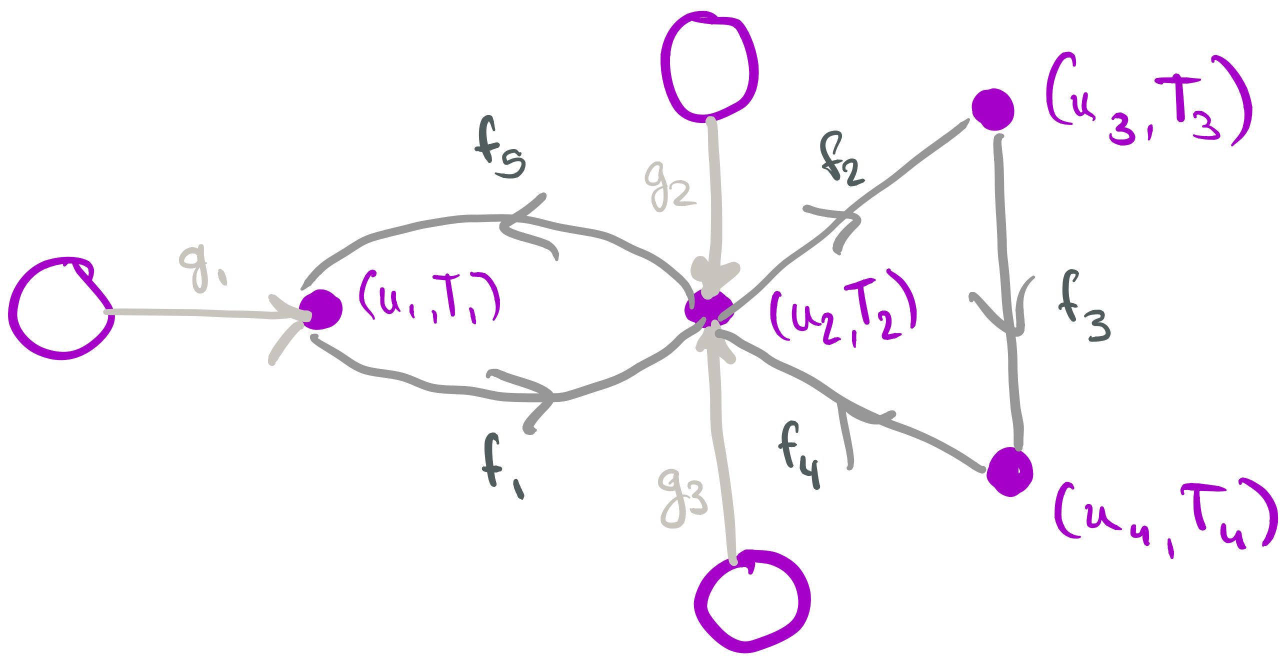
- $ob(\mathcal{P})$ is the collection of states of the thermodynamic system
- $ar(\mathcal{P})$ is the collection of physical processes

Lawvere assumes there is given a functor from $\mathcal{P}$ to the additive monoid of extended reals, which he calls the entropy supply, called s.

processes either consume (+) or generate (−) entropy.

Heat Networks

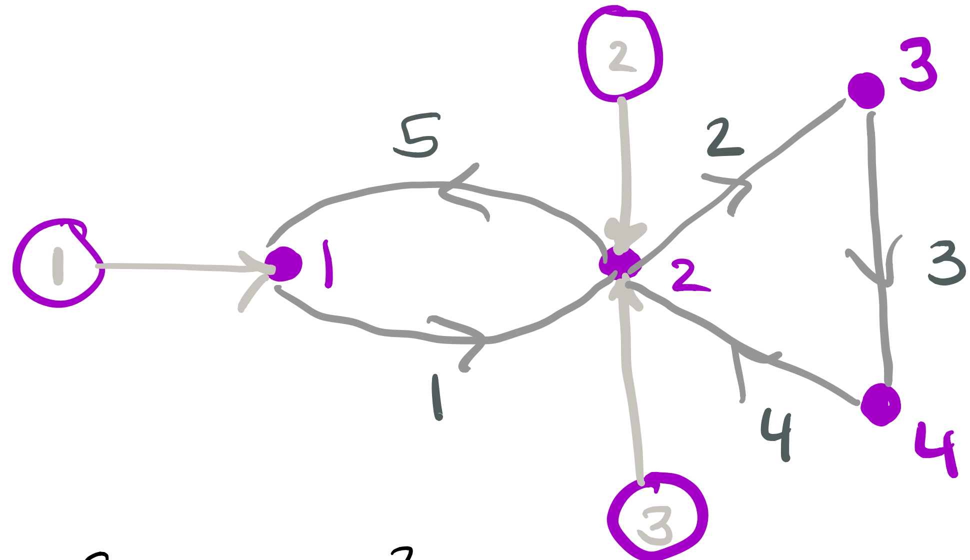
Lawvere doesn't provide an example of p or s, so we set out to develop one.



Entropy supply will be related to flows through the interfaces of such a network.

Open Graphs as Heat Networks

$$X \xrightarrow{\iota} V \underset{tgt}{\overset{src}{\leftleftarrows}} E$$



$X = \{1, 2, 3\}$

$1 \xmapsto{\iota} 1 \qquad 2 \xmapsto{\iota} 2$

$3 \xmapsto{\iota} 2$

$V = \{1, 2, 3, 4\}$

$E = \{1, 2, 3, 4, 5\}$

$1 \xmapsto{(src, trg)} (1, 2) \qquad 2 \xmapsto{(src, trg)} (2, 3)$

$3 \xmapsto{(src, trg)} (3, 4) \qquad 4 \xmapsto{(src, trg)} (4, 2)$

$5 \xmapsto{(src, trg)} (2, 1)$

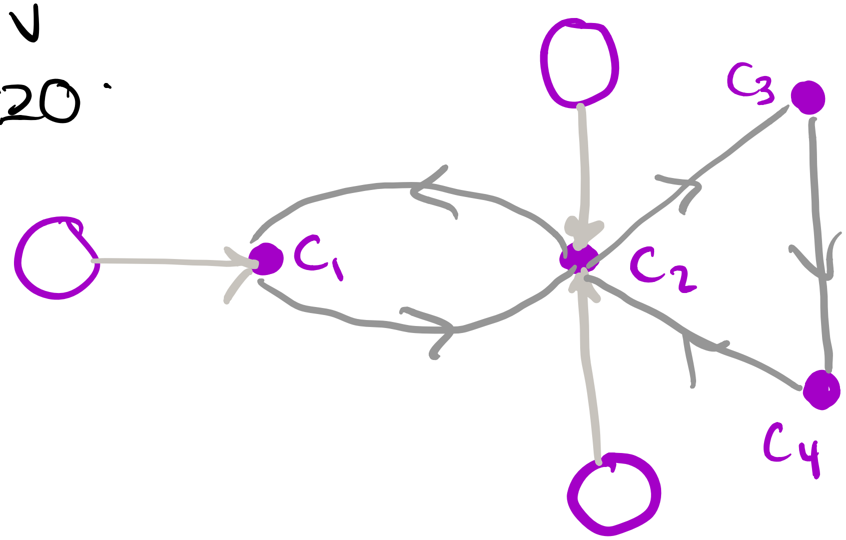
Open Graphs as Heat Networks, II

The open graph is equipped with a function $C: V \to \mathbb{R}_{\geq 0}$ that assigns a heat capacity to each vertex.

A state of the heat network is specified by a function $u: V \to \mathbb{R}_{\geq 0}$ that assigns an energy to each vertex.

The phase space is $\mathbb{R}_{\geq 0}^{V}$.

The temperature of a vertex $v \in V$ at a certain state is $T_v = u_v / c_v$.



Processes in Heat Networks

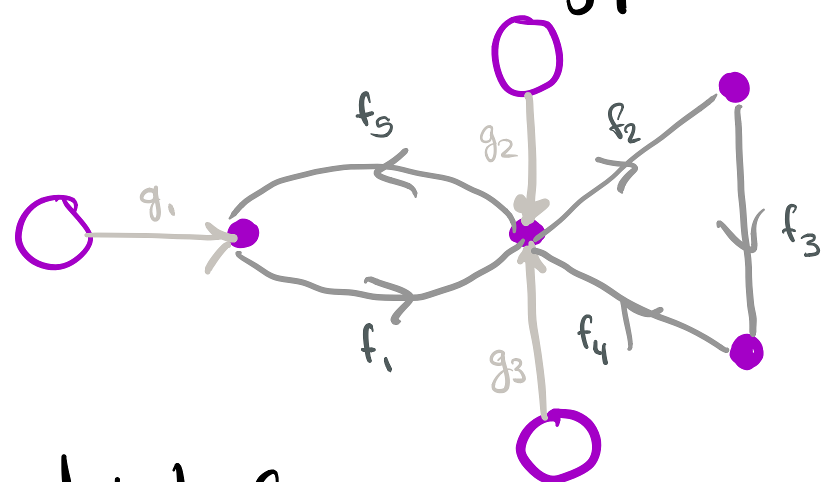
A process consists of:

- a time interval $[0, \tau]$
- a piecewise-differentiable function $U : [0, \tau] \to \mathbb{R}_{\geq 0}^V$ assigning internal energy
- integrable functions

  $f : [0, \tau] \longrightarrow \mathbb{R}^E$

  $g : [0, \tau] \longrightarrow \mathbb{R}^X$

  assigning heat fluxes to the internal edges and interfaces



such that...

Constraints on Processes

$\forall v \in V$. almost all $t \in [0, \tau]$.

- $U'_v(t) = \sum_{tgt(e)=v} f_e(t) - \sum_{src(e)=v} f_e(t) + \sum_{\iota(x)=v} g_x(t)$

  Energy is conserved.

- $f_e(t) \geq 0$ if $T_{src(e)} > T_{trg(e)}$
  $f_e(t) \leq 0$ if $T_{src(e)} < T_{trg(e)}$

  Heat only flows from hot to cold.

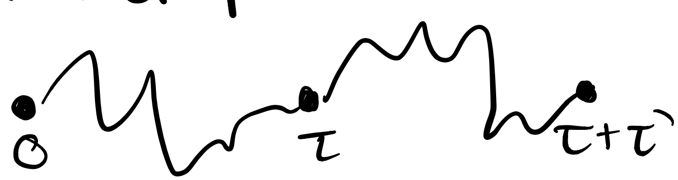
Category of Heat Network Processes

Define a category HeatProc:

- ob(HeatProc) is the collection of states

  $u \in \mathbb{R}^V_{\geq 0}$

- ar(HeatProc) is the collection of processes.
  The domain of a process is the initial state and the codomain is the final state.
- identity morphisms are "freeze" processes, which have duration zero and zero heat flow
- composition of processes is concatenation



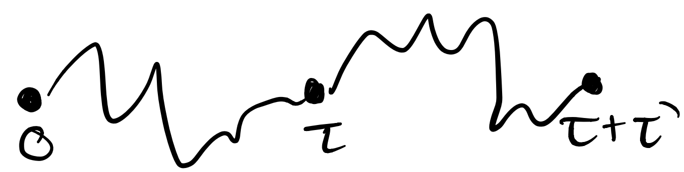
Entropy Supply Functor

Lawvere assumes an entropy supply functor $\mathcal{S} : \mathcal{P} \longrightarrow B(\overline{\mathbb{R}}, +)$.

In thermodynamics, $\left(\delta S = \frac{\delta Q}{T}\right) \Rightarrow \left(\Delta S = \int \frac{\delta Q}{T}\right)$

We define the entropy supply functor for a process P in **HeatProc** as $\mathcal{S}(P) := \int_0^{\tau} \sum_{x \in X} \frac{g_x(t)}{T_{L(x)}(t)} \, dt$

- $\mathcal{S}$ preserves identity morphisms
  $\mathcal{S}(id_u) = 0$
- $\mathcal{S}$ preserves composition
  $\mathcal{S}(P \cdot P') = \mathcal{S}(P) + \mathcal{S}(P')$



Naive Clausius Property

The Clausius theorem is $\oint \frac{\delta Q}{T_{surr}} \leq 0$, which is the statement that thermodynamic cycles cannot decrease entropy.

Lawvere defines the naive Clausius property for entropy supplies to be $s(P: u \to u) \leq 0$.

Naive Clausius Property, II

Theorem. $S : \mathrm{HeatProc} \to B(\overline{\mathbb{R}}, +)$ satisfies the naive Clausius property.

$$S(P) = \int_0^\tau \sum_{x \in X} \frac{g_x(t)}{T_{\iota(x)}(t)}\, dt$$

(conservation of energy)

$$= \underbrace{\int_0^\tau \sum_{v \in V} \frac{U'_v(t)}{T_v(t)}\, dt}_{=0 \text{ (cyclicity)}} - \underbrace{\int_0^\tau \sum_e f_e(t) \left(\frac{1}{T_{tgt(e)}(t)} - \frac{1}{T_{src(e)}(t)} \right) dt}_{\leq 0 \text{ (heat flows from hot to cold)}}$$

$$S(P : u \to u) \leq 0 \checkmark$$

Future Directions

- adapting network model to other thermodynamic fluxes (work, particles)

- investigating functoriality of construction of HeatProc from open graphs