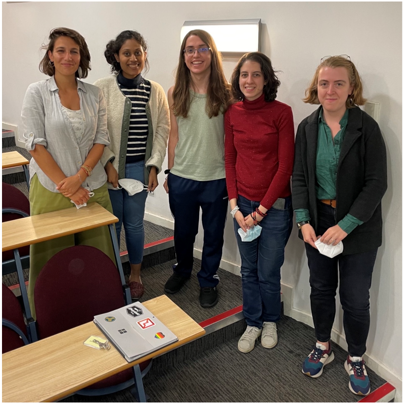
Fuzzy Type Theory for Opinion Dynamics

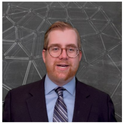
Shreya Arya, Greta Coraglia, Paige North,
Sean O'Connor, Hans Riess, Ana Tenório

ACT

2022

Our group





Left to right: Greta, Shreya, Sean, Ana, Paige, Hans

Fuzzy Type Theory for Opinion Dynamics

Modeling Opinions

Want to model ”proof-relevant opinions”
We need

- types as opinions
- terms as proofs of/reasons for opinions
- fuzzy logic as confidence/certainty/strength of opinions

Type Theories and Fuzzy Logic

Enriching over a different monoidal category gives us a different type theory/logic

	binary	*fuzzy*
propositions	$\{0, 1\}$	$[0, 1]$
types	**Set**	$\Sigma_{S:\mathbf{Set}}\, S \to [0, 1]$

Categories and Type Theories

$$\textit{Type theories} \rightleftarrows \textit{Categories}$$

Given a type theory we can obtain a category where:

- the objects are contexts Γ
- the morphisms are (lists of) terms

Categorical Semantics

types in context	a class of maps (*projections*)
$\Gamma \vdash A\ \mathtt{type}$	$\Gamma.A \xrightarrow[p_A]{} \Gamma$
terms	sections of projections
$\Gamma \vdash a : A$	$\Gamma.A \underset{p_A}{\overset{a}{\leftrightarrows}} \Gamma$
substitution	pullback along projections
...	...

Enriched Categories and Fuzzy Types

Our strategy: **enrich the categories, read the type theory!**

Call $V = \Sigma_{S:\mathbf{Set}} S \to [0,1]$ the category whose

- objects are pairs $(S, |\_|_S)$ with S a set and $|\_|_S : S \to [0,1]$ a function, called *valuation*
- morphisms $f : (S, |\_|_S) \to (T, |\_|_T)$ are order-preserving functions between S and T

$$\textit{Fuzzy type theories} \rightleftarrows V\textit{- Categories}$$

Intuition

a V-category $\mathcal{C}$	an agent in the system
a context	a set of beliefs
a type (in context)	a belief (and its premises)
a term of type A	a proof of the belief A

- we want definite beliefs $\Rightarrow$ non-fuzzy types
- but their reasons might be subject to uncertainty $\Rightarrow$ fuzzy terms

Fuzzy Type Theory for Opinion Dynamics

The idea

The math

Projections and Sections

Axiom: Types are not fuzzy

For all A, $|\, p_A \,|_{\underline{\mathsf{hom}}(\Gamma.A,\Gamma)} = 1$.

Normally, terms are sections of projections, but

$$\begin{tikzcd} \Gamma \arrow[r, "s"'] \arrow[rr, bend left, "id"] & \Gamma.A \arrow[r, "p_A"'] & \Gamma \end{tikzcd}$$

$|\, id \,| = 1 \implies |\, p_A \,| \cdot |\, s \,| = 1 \implies |\, p_A \,| = |\, s \,| = 1$
This is too much of a restriction for us!

α-SECTIONS

Definition: α-sections

We say s is a α-section of p if $p \circ s = id$ as functions and $|p| \cdot |s| \geq \alpha$

$$\text{Denoted} \quad \Gamma \vdash s :_\alpha A$$

$$\text{and we have} \quad \frac{\Gamma \vdash s :_\alpha A}{\Gamma \vdash s :_\beta A} \quad \text{for all } \beta \leq \alpha$$

Substitution and Pullbacks

Classically, substitution is performed as pullback along projections. Problem is that in the enriched case we need to consider weighted pullbacks!

Weighted pullbacks are a special case of weighted limits, which replace limits in enriched settings.

This can be used to determine the universal property of weighted pullbacks in V-categories

Weighted Pullbacks

Consider a pullback in **Set**-categories

$$\begin{tikzcd} A \times_C B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

What in **Set**-categories is the bijection

$$\mathsf{hom}(Z, A \times_C B) \cong \mathsf{hom}(Z, A) \times_{\mathsf{hom}(Z,C)} \mathsf{hom}(Z, B)$$

can be viewed as a weighted pullback in which

$$\mathsf{hom}(Z, A \times_C B) \cong \mathsf{hom}(Z, A)^{\mathbb{1}} \times_{\mathsf{hom}(Z,C)^{\mathbb{1}}} \mathsf{hom}(Z, B)^{\mathbb{1}}$$

With this perspective, we say that a regular pullback is $(\mathbb{1}, \mathbb{1}, \mathbb{1})$-weighted, with $\mathbb{1} = \{*\}$

FUZZY SUBSTITUTION I

We need to find reasonable weights!

- $(\mathbb{1}, \mathbb{1}, \mathbb{1})$ with $\mathbb{1}$ the terminal object doesn't work with fuzzy terms
- We can denote $\mathbb{1}_x = (\{*\}, \mathit{const}(x))$ to use as our weights
- $(\mathbb{1}_{\text{val(-)}}, \mathbb{1}, \mathbb{1}_{\text{val(-)}})$

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "\geq\min\{1,\delta/\alpha,\gamma/\beta\}"'] \arrow[drr, bend left, "\delta"] \arrow[ddr, bend right, "\gamma"'] & & \\
& S \times_R T \arrow[r, "\geq\alpha"] \arrow[d, "\geq\beta"'] & T \arrow[d, "\beta"] \\
& S \arrow[r, "\alpha"'] & R
\end{tikzcd}
$$

Something weird

$$
\begin{tikzcd}
\mathbb{1} \arrow[dr, dashed] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "1"'] & & & \\
& {\color{red}\mathbb{1}.A} \arrow[r] \arrow[d] & \mathbb{1}.A.A \arrow[r] \arrow[d] & \mathbb{1}.A \arrow[d] \\
& \mathbb{1} \arrow[r, "\alpha"] & \mathbb{1}.A \arrow[r] \arrow[u, bend right, "1"'] & \mathbb{1}
\end{tikzcd}
\qquad
\frac{\vdash x :_{\alpha} A}{\vdash y :_{1} A}
$$

But the top-left ${\color{red}\mathbb{1}.A}$ is obtained by pullback along a map of value α, so it isn't the same object at $\mathbb{1}.A$.

Resolution

$$
\begin{tikzcd}
\mathbb{1} \arrow[dr, dashed] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "1"'] & & & \\
& \mathbb{1}.A_\alpha \arrow[r] \arrow[d] & \mathbb{1}.A.A \arrow[r] \arrow[d] & \mathbb{1}.A \arrow[d] \\
& \mathbb{1} \arrow[r, "\alpha"] & \mathbb{1}.A \arrow[r] \arrow[u, bend right, "1"'] & \mathbb{1}
\end{tikzcd}
\qquad
\frac{\vdash x :_\alpha A}{\vdash y :_1 A}
$$

We can denote the top-left $\mathbb{1}.A$ as $\mathbb{1}.A_\alpha$ and we can read

$$\frac{\vdash s :_\alpha A}{\vdash t :_1 A_\alpha}$$

as "Given a proof of A with confidence α, we can prove with confidence 1 that we can prove A with confidence α".

Fuzzy substitution II

$$\frac{\Gamma, x : A, \Delta \vdash t :_\beta B \quad \Gamma \vdash u :_\alpha A}{\Gamma, \Delta[u/x] \vdash t[u/x] :_\beta B[u/x]_\alpha}$$

$$\begin{tikzcd}
\Gamma.\Delta \arrow[r, dashed, "\beta"] \arrow[dr, "1"'] \arrow[rr, bend left, "\alpha\cdot\beta"] & \Gamma.\Delta.B_\alpha \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.\Delta.B \arrow[d] \\
& \Gamma.\Delta \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.\Delta \arrow[u, bend right, "\beta"'] \arrow[d] \\
& \Gamma \arrow[r, "\alpha", "u"'] & \Gamma.A \arrow[d] \\
& & \Gamma \arrow[u, bend right, "\alpha"']
\end{tikzcd}$$

Validity

Theorem

Such a V-category satisfies (a fuzzy version of) all structural rules of (not-yet-dependent) type theory.

Therefore we have categories to encode the logical system of the agents in our system.

The Dynamic

- The work of Jakob Hansen and Robert Ghrist uses a cellular sheaf $F : Inc(G) \to Vect$ to study opinion dynamics
- The work of Hans Riess and Robert Ghrist studies cellular sheaves of the form $F : Inc(G) \to Lattices$
- We want to explore $F : Inc(G) \to V\text{-}Cat$

Future Work

- Give an enriched categorical interpretation for the dependent fuzzy types (and address definitional equality);
- Replace [0,1] by any ordered monoid M;
- Explore the dynamic side

References I

Ghrist, R., and Riess, H.
Cellular sheaves of lattices and the Tarski laplacian.
Homology, Homotopy and Applications 24, 1 (2022), 325–345.

Hansen, J., and Ghrist, R.
Opinion dynamics on discourse sheaves, 2020.

Hofmann, M.
Syntax and semantics of dependent types.
In *Extensional Constructs in Intensional Type Theory*. Springer, 1997, pp. 13–54.

Thank you!