Catergorical Causal Models

with intervention 2-cells, cycles & compositionality

Adjoint school group 3

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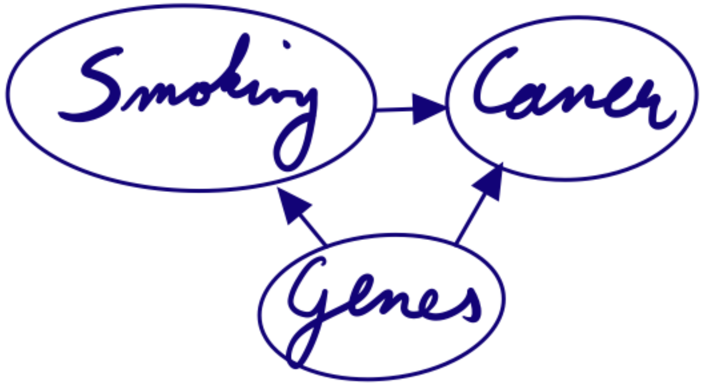
Bayesian Causal Networks

- Directed Acyclic Graph $\mathrm{G} = (\mathcal{V}, \mathcal{E})$
- For each vertex $v \in \mathcal{V}$, measurable space X_v
  - Assumed finite
- For each vertex, a conditional probability distribution

$$P(X_v | \prod_{p \in Pa(v)} X_p)$$

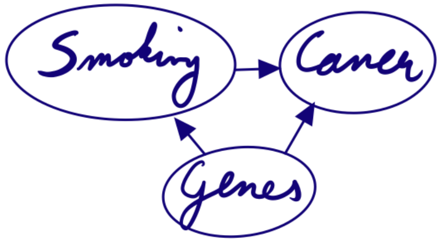
Example:

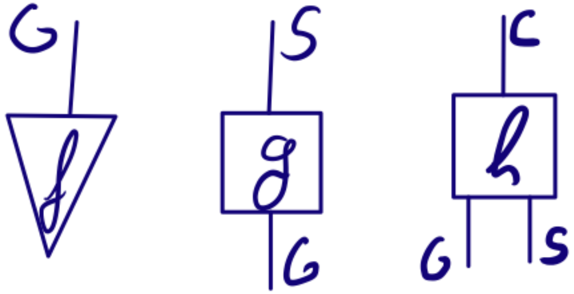


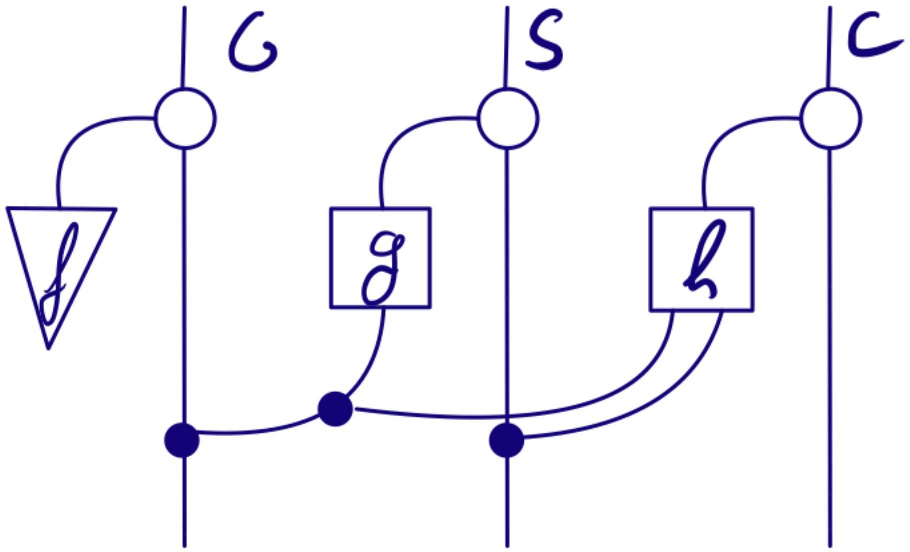
- Binary
- $P(G), P(S|G), P(C|S, G)$

Syntax Category

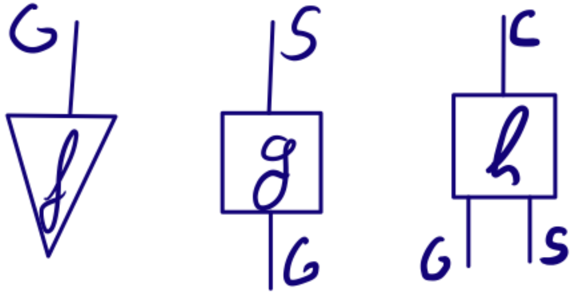
- Given DAG $\mathrm{G} = (\mathcal{V}, \mathcal{E})$
- Syntax category Syn_{G}
  - Symmetric Monoidal
  - Objects (tensor products of) $v \in \mathcal{V}$
  - Discrete Markov Restriction
  - Morphism from parents for each vertex
  - Graph encoded as endomorphism

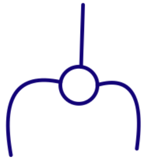




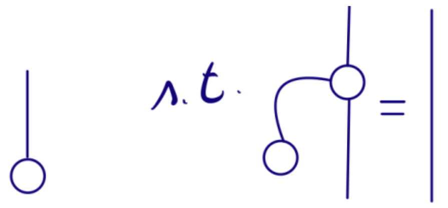


Semantics functor $F: Syn_{\mathrm{G}} \to Kl(D_{\leq 1})$

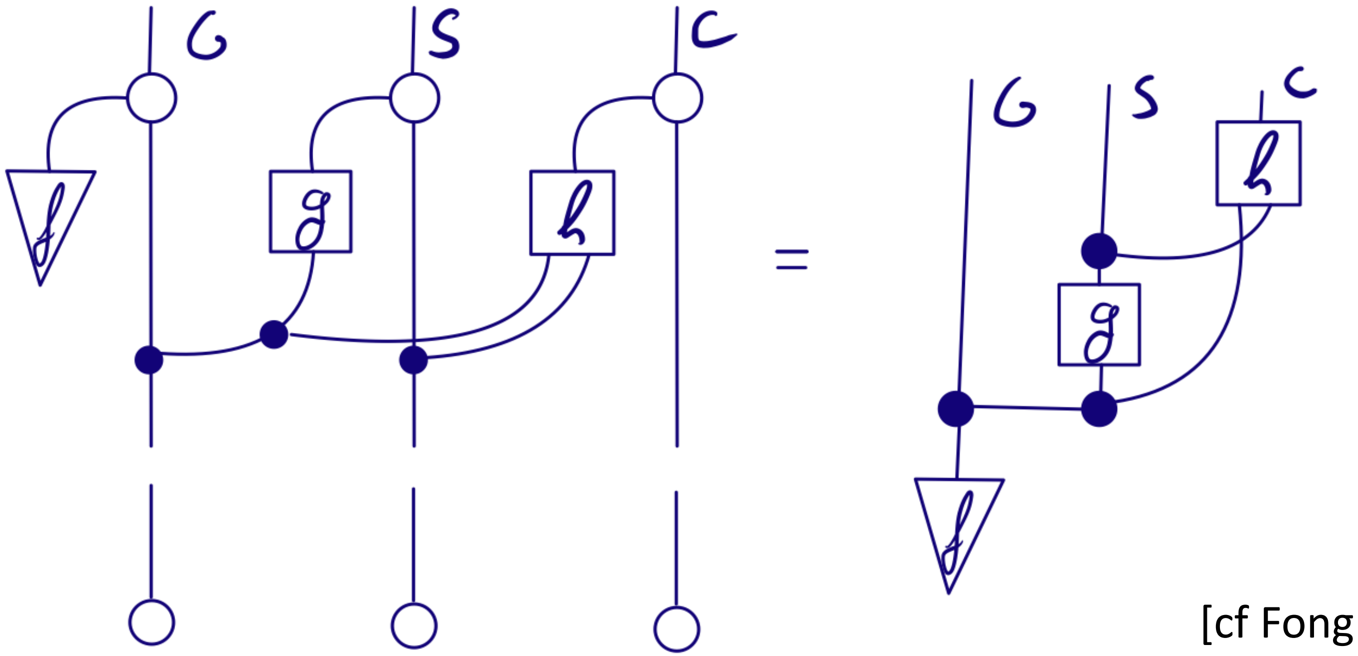
- $Kl(D_{\leq 1})$ has:
  - Objects: measurable spaces
  - Morphisms: Markov kernels to measures (not necessarily probability measures)
- F maps object v to measurable space X_v
- F maps morphisms 

to $P(G), P(S|G), P(C|S,G)$
- F maps comparator  to $P(X = x|X_1 = x_1, X_2 = x_2) = [x = x_1 = x_2]$

Solution



- Embed Syn_{G} in compact closed category $\overline{Syn_{\mathrm{G}}}$, including monoid unit
- Compact closure for $Kl(D_{\leq 1})$ is $Mat(\mathbb{R}_{\geq 0})$ of non-negative matrices, with unit $(1, 1, \dots)$
- Obtain solution by precomposition with unit

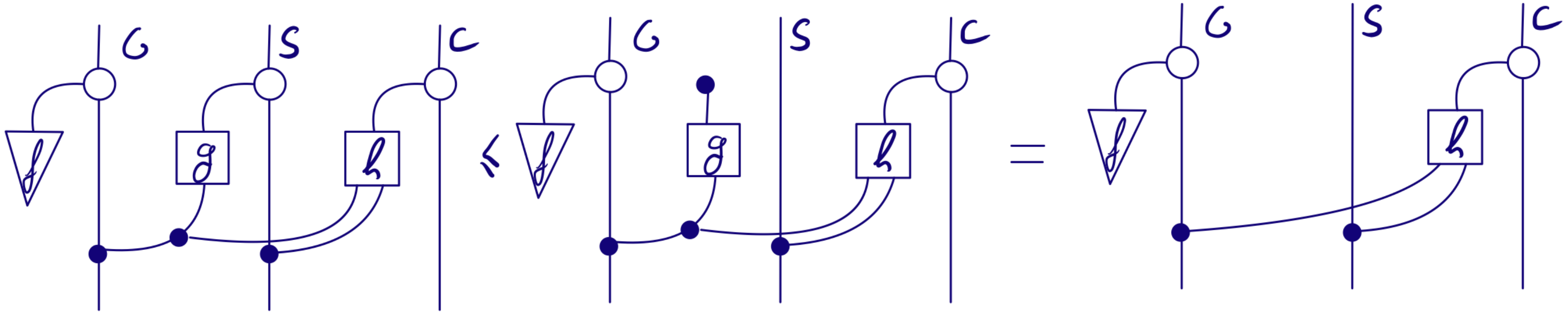


[cf Fong & Spivak 2013]

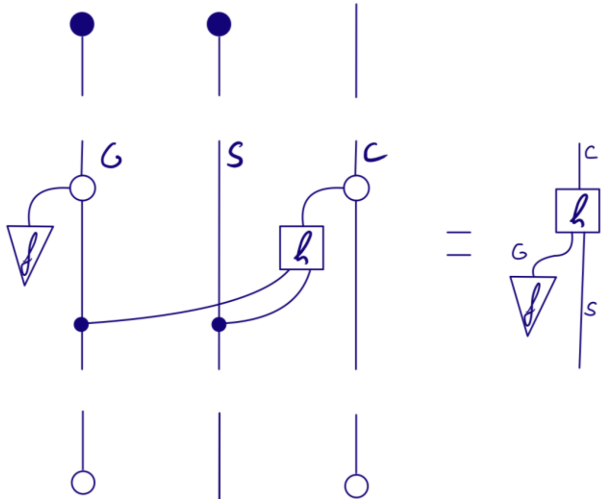
Intervention 2-cells

- Enrich Syn_{G} with poset generated by

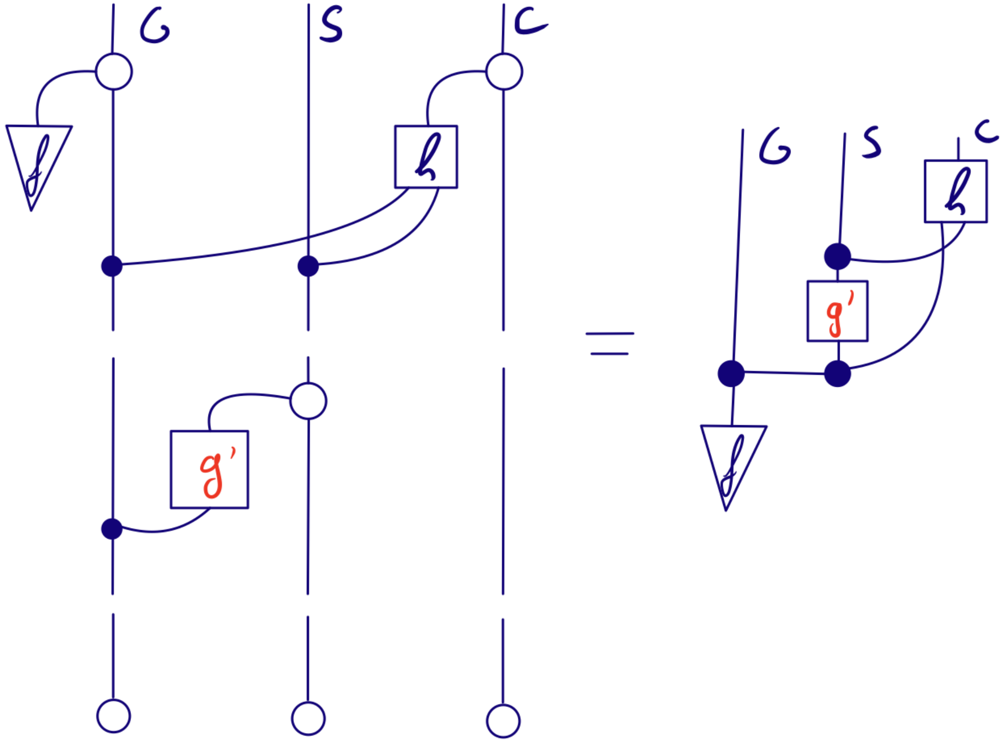
- Allows for interventions



- Solution with input S and output C
  - In causal notation written as $P(C|do(S))$

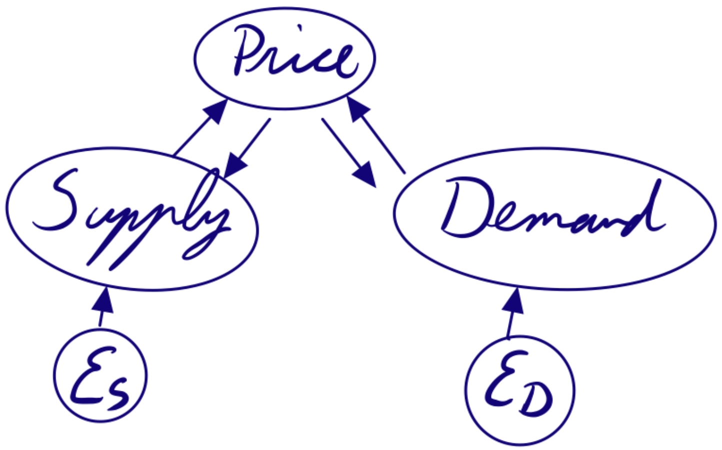


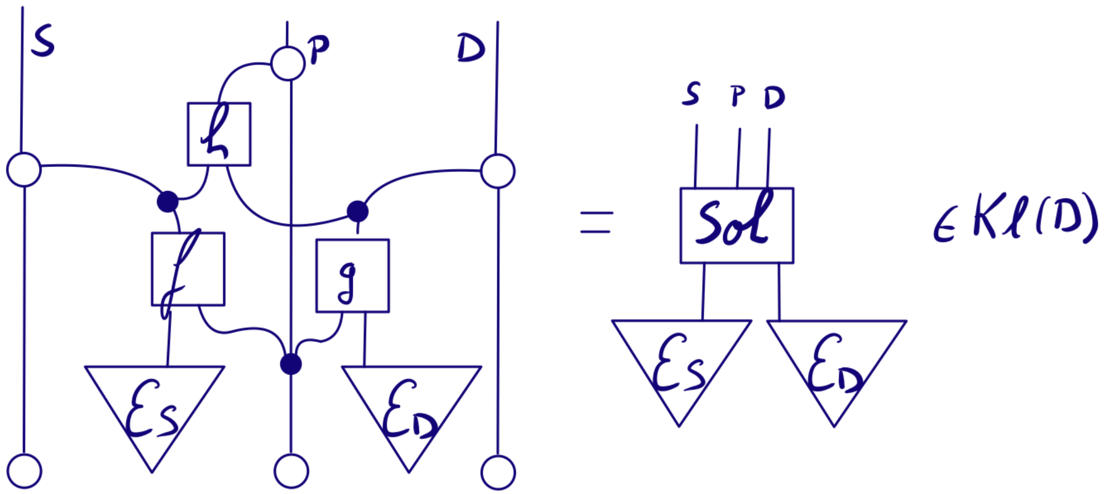
Compositionality



- Corresponds to soft intervention on S, changing conditional $P(S|G)$

Cycles





- RHS is a solution to cyclic model if it does not contain comparator or monoid unit
- For deterministic maps f, g, h: uniform distribution over fixpoints

Open questions

- Implement DRM categories via probabilistic programming
  - How to take conditionals, possibly via normalization
- Formalize notion of solution
- Generalize to infinite spaces
- Bicategorical structures on stochastic maps
- Do-calculus & causal inference
  - Cf. Jacobs et al “Causal Inference via String diagram Surgery” (2019)