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Quantale Enriched Framework for Mathematical Morphology

Ignacio Bellas Acosta[1]

[1]School of Computer Science
University of Leeds

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Overview

Background

- Mathematical Morphology (MM), 1960s image processing (Serra [2], Serra et al [3]).
- Two main ingredients:
  - Binary image: Modelled as $I \subseteq \mathbb{Z}^2$.
  - Structuring element: A pattern of of pixels $B \subseteq \mathbb{Z}^2$.

Dilation

$I \oplus B := \bigcup_{b \in B} I_b$ where $I_b := \{i + b \mid i \in I\}$.

Erosion

$I \ominus B := \bigcap_{b \in B} I_{-b}$ where $I_{-b} := \{i - b \mid i \in I\}$.

Example

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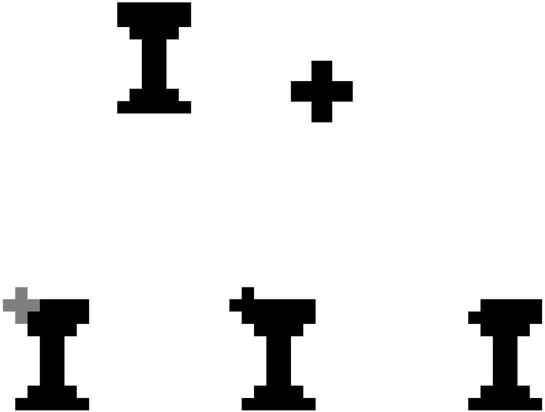

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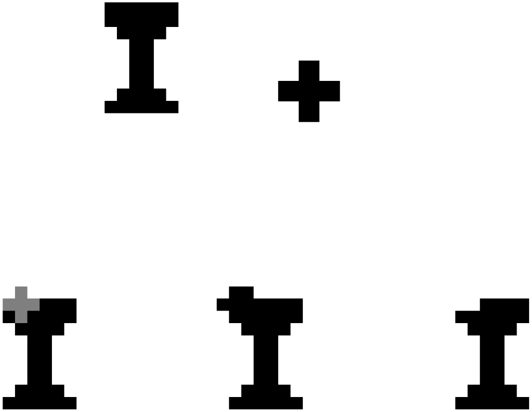

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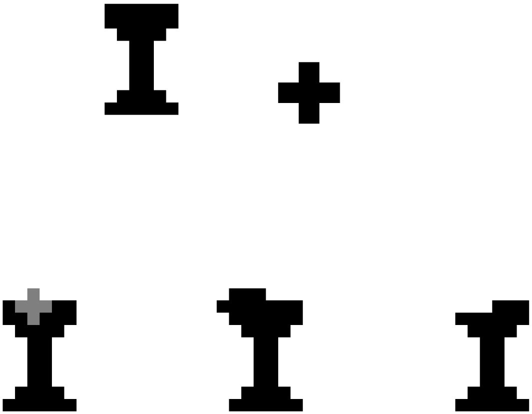

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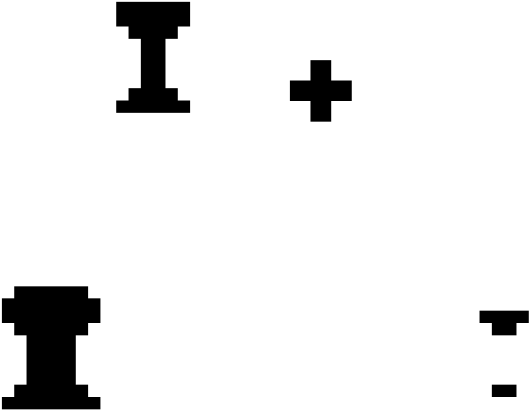

Relational Approach

- Structuring elements $B \subseteq \mathbb{Z}^2$ induce binary relations $R_B := \{(x, x+b) \mid x \in \mathbb{Z}^2 \text{ and } b \in B\}$.

Dilation of I by R

$$I \oplus R := \{x \in \mathbb{Z}^2 \mid \exists y \in \mathbb{Z}^2 : yRx \land y \in I\}$$

Erosion of I by R

$$I \ominus R := \{x \in \mathbb{Z}^2 \mid \forall y \in \mathbb{Z}^2 : xRy \to y \in I\}$$

- $I \oplus B = I \oplus R_B$,
- $I \ominus B = I \ominus R_B$,
- $I \oplus \breve{R} = \Diamond I$
- $I \ominus R = \Box I$

Relational approach

Equipping $\mathcal{P}\mathbb{Z}^2$ with the binary relation $\subseteq$ yields an adjunction:

Corollary

For any $R \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$\mathcal{P}\mathbb{Z}^2 \underset{-\ominus R}{\overset{-\oplus R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}\mathbb{Z}^2$$

Lemma

$$\begin{array}{ccc} \mathcal{P}\mathbb{Z}^2 & \xrightarrow{-\oplus R} & \mathcal{P}\mathbb{Z}^2 \\ {\scriptstyle -}\uparrow & & \downarrow{\scriptstyle -} \\ (\mathcal{P}\mathbb{Z}^2)^{op} & \xrightarrow{-\ominus \check{R}} & (\mathcal{P}\mathbb{Z}^2)^{op} \end{array} \qquad \begin{array}{ccc} \mathcal{P}\mathbb{Z}^2 & \xrightarrow{-\ominus R} & \mathcal{P}\mathbb{Z}^2 \\ {\scriptstyle -}\uparrow & & \downarrow{\scriptstyle -} \\ (\mathcal{P}\mathbb{Z}^2)^{op} & \xrightarrow{-\oplus \check{R}} & (\mathcal{P}\mathbb{Z}^2)^{op} \end{array}$$

Graph MM

- Binary Graph MM (Stell [4]):

Concept	(Binary) Set MM	(Binary) Graph MM
Grid of Pixels	Set $\mathbb{Z}^2$	
Relation	$R \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$	
Image space	$\mathcal{P}\mathbb{Z}^2$	
Converse	$\breve{R} \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$	
Complement	$- : \mathcal{P}\mathbb{Z} \to (\mathcal{P}\mathbb{Z})^{op}$	
Dilation	$- \oplus R : \mathcal{P}\mathbb{Z} \to \mathcal{P}\mathbb{Z}$	
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- Semantics for BISKT (Stell et al [5]):

  - $\blacklozenge\varphi := [\![\varphi]\!] \oplus R$
  - $\Box\varphi := [\![\varphi]\!] \ominus R$
  - $\Diamond\varphi \leftrightarrow \lrcorner\Box\neg\varphi$,
  - $\Diamond\varphi := [\![\varphi]\!] \oplus \circlearrowleft R$
  - $\blacksquare\varphi := [\![\varphi]\!] \ominus \circlearrowleft R$
  - $\blacksquare\varphi \leftrightarrow \neg\blacklozenge\lrcorner\varphi$

Goal

Extend the MM framework to account for:

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- Graphical images

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Extend the MM framework to account for:

- Graphical images $\rightarrow$ Order Theory

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- Images valued on Greyscale/Colours

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- Images valued on Greyscale/Colours$\rightarrow$ Quantale Theory

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Order Theory $+$ Quantale Theory $\rightarrow$ Quantale Enriched Theory

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Extend the MM framework to account for:

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Order Theory + Quantale Theory $\rightarrow$ Quantale Enriched Theory

Project

We propose a framework within Quantale Enriched Category Theory that accounts for Colour/Greyscale Graph MM.

Quantales

Quantale $\mathcal{Q}$

A complete lattice $\mathcal{Q}_0$ equipped with a composition operation $\cdot : \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$ that preserves sups in both arguments and has a unit element $e \in \mathcal{Q}_0$.

There are two residual operations $\rhd : \mathcal{Q} \times \mathcal{Q}^{co} \to \mathcal{Q}^{co}$ and $\lhd : \mathcal{Q}^{co} \times \mathcal{Q} \to \mathcal{Q}^{co}$ satisfying the following condition:

$$f \cdot - \dashv f \rhd - \text{ and } - \cdot g \dashv - \lhd g.$$

Quantale morphism

A quantale morphism $\alpha : \mathcal{Q} \to \mathcal{Q}'$ is a sup-lattice morphism that preserves the composition and the unit element.

Involutive and Girard Quantales

Involutive Quantale

A quantale $\mathcal{Q}$ equipped with an involutive quantale morphism $(-)^{\dagger} : \mathcal{Q} \to \mathcal{Q}^{op}$ is said to be an involutive quantale.

Girard Quantale

A quantale $\mathcal{Q}$ is Girard if there exists an element $d \in \mathcal{Q}_0$ that:

- $f \rhd d = d \lhd f$ (Cyclic),
- $d \lhd (f \rhd d) = f$ (Dualising)

for every $f \in \mathcal{Q}_0$.

The cyclic and dualising element in a Girard quantale $\mathcal{Q}$ induces an involutive sup-lattice morphism $(-)^{\perp} : \mathcal{Q} \to \mathcal{Q}^{coop}$ ($f \mapsto f \rhd d$).

Examples

Some examples of involutive Girard quantales:

- The Boolean algebra 2 where $\cdot = \wedge$.
- The three element chain 3 equipped with the composition operation:

$\cdot$	$\top$	1	$\bot$
$\top$	$\top$	$\top$	$\bot$
1	$\top$	1	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

- The diamond lattice M_3 ($\bot \leq a, b, c \leq \top$) equipped with the composition operation:

$\cdot$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	b	c	$\top$
b	$\bot$	b	c	a	$\top$
c	$\bot$	c	a	b	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

$\mathcal{Q}$-enriched structures

$\mathcal{Q}$-category

A $\mathcal{Q}$-enriched category $\mathcal{X}$ consists of a set $\mathcal{X}_0$ equipped with a function $\mathcal{X}(-,-) : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathcal{Q}$ that satisfies:

- $e \leq \mathcal{X}(x,x)$,
- $\mathcal{X}(x,x') \cdot \mathcal{X}(x',x'') \leq \mathcal{X}(x,x'')$.

$\mathcal{X}(-,-)$ induces a preorder on $\mathcal{X}_0$:

- $x \leq x'$ (in $\mathcal{X}$) $\Leftrightarrow e \leq \mathcal{X}(x,x')$ (in $\mathcal{Q}$).

$\mathcal{Q}$-functor

A $\mathcal{Q}$-functor $F : \mathcal{X} \to \mathcal{Y}$ is a function where $\mathcal{X}(x,x') \leq \mathcal{Y}(Fx, Fy)$.

We let $\mathbf{Cat}_{\mathcal{Q}}$ be the 2-category of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors.

$\mathcal{Q}$-distributors

$\mathcal{Q}$-distributor

Given two $\mathcal{Q}$-categories $\mathcal{X}$ and $\mathcal{Y}$, a $\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{Y}$ is a function $R : \mathcal{X}_0 \times \mathcal{Y}_0 \to \mathcal{Q}$ satisfying the following two axioms:

- $\mathcal{X}(x, x') \cdot R(x', y) \leq R(x, y)$
- $R(x, y) \cdot \mathcal{Y}(y, y') \leq R(x, y')$

Let $\mathbf{Dist}_{\mathcal{Q}}$ be the quantaloid of $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors. Then:

- $(R \bullet S)(x, z) := \bigvee_{y \in \mathcal{Y}_0} R(x, y) \cdot S(y, z)$,
- $(R \blacktriangleright T)(y, z) := \bigwedge_{x \in \mathcal{X}_0} R(x, y) \triangleright T(x, z)$,
- $(T \blacktriangleleft S)(x, y) := \bigwedge_{z \in \mathcal{Z}_0} T(x, z) \triangleleft S(y, z)$.

$\mathcal{Q}$(-co)-presheaves

$\mathcal{Q}$-co-presheaves

A $\mathcal{Q}$-co-presheaf is a $\mathcal{Q}$-distributor $\varphi : * \nrightarrow \mathcal{X}$. $\mathcal{UX}$ is the $\mathcal{Q}$-category of co-presheaves where $\mathcal{UX}(\varphi, \varphi') = \varphi \blacktriangleleft \varphi'$.

$$\varphi \leq \varphi' \text{ (in } \mathcal{UX}) \text{ iff } \varphi' \leq \varphi \text{ (in } \mathbf{Dist}_{\mathcal{Q}})$$

$\mathcal{Q}$-presheaves

A $\mathcal{Q}$-presheaf is a $\mathcal{Q}$-distributor $\psi : \mathcal{X} \nrightarrow *$. $\mathcal{DX}$ is the $\mathcal{Q}$-category of presheaves where $\mathcal{DX}(\psi, \psi') = \psi \blacktriangleright \psi'$.

$$\psi \leq \psi' \text{ (in } \mathcal{DX}) \text{ iff } \psi \leq \psi' \text{ (in } \mathbf{Dist}_{\mathcal{Q}})$$

Example

- Any preorder $\mathbb{X} = (X, \leq)$ is a $\mathcal{Q}(2)$-category $\mathcal{X}$ where $\mathcal{X}(x, x') = \top$ iff $x \leq x'$,

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- Any monotone relation $R \subseteq \mathbb{X}^{op} \times \mathbb{X}$ is a $\mathcal{Q}(2)$-distributor $R : \mathcal{X} \nrightarrow \mathcal{X}$.

Example

- Any preorder $\mathbb{X} = (X, \leq)$ is a $\mathcal{Q}(2)$-category $\mathcal{X}$ where $\mathcal{X}(x, x') = \top$ iff $x \leq x'$,
- Any monotone relation $R \subseteq \mathbb{X}^{op} \times \mathbb{X}$ is a $\mathcal{Q}(2)$-distributor $R : \mathcal{X} \nrightarrow \mathcal{X}$.
- Any upset $U \in \mathrm{Up}(\mathbb{X})$ is a $\mathcal{Q}(2)$-co-presheaf $\varphi_U : * \nrightarrow \mathcal{X}$ where $\varphi_U(x) = \top$ iff $x \in U$.

Example

Concept	Graph MM	$\mathcal{Q}$-generalisation
Grid	$\mathbb{X}$	$\mathcal{Q}$-category $\mathcal{X}$
Relation	$R \subseteq \mathbb{X}^{op} \times \mathbb{X}$	$\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{X}$
Image Space	$\mathrm{Up}(\mathbb{X})$	$\mathcal{Q}$-category $\mathcal{U}\mathcal{X}$
Dilation	$- \oplus R$	
Erosion	$- \ominus R$	
Converse	$\smile R \subseteq \mathbb{X}^{op} \times \mathbb{X}$	
Complement	$\neg, \lrcorner$	

Generalising dilations and erosions

Theorem (Stubbe [6])

For any two $\mathcal{Q}$-categories $\mathcal{X}$ and $\mathcal{Y}$, $\mathbf{Dist}_{\mathcal{Q}}(\mathcal{X},\mathcal{Y})$ is locally equivalent to $\mathbf{Co\text{-}Cont}^{op}_{\mathcal{Q}}(\mathcal{U}\mathcal{X},\mathcal{U}\mathcal{Y})$ and $\mathbf{Cont}^{co}_{\mathcal{Q}}(\mathcal{U}\mathcal{X},\mathcal{U}\mathcal{Y})$.

Idea

Any $\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{Y}$ acts on $\mathcal{U}\mathcal{X}$ and $\mathcal{U}\mathcal{Y}$ defining an adjunction of $\mathcal{Q}$-categories:

$$\mathcal{U}\mathcal{Y} \underset{-\bullet R}{\overset{-\blacktriangleleft R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{U}\mathcal{X}$$

- $- \bullet R =$ $\mathcal{Q}$-generalisation of $- \oplus R$
- $- \blacktriangleleft R =$ $\mathcal{Q}$-generalisation of $- \ominus R$

Defining the converse

Lemma

For any $\mathcal{Q}$-category $\mathcal{X}$, the structure $\mathcal{X}^\dagger$ where $\mathcal{X}^\dagger(x, x') = \mathcal{X}(x', x)^\dagger$ is a $\mathcal{Q}$-category. For any $\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{Y}$, the function $R^\dagger(y, x) = R(x, y)^\dagger$ is a $\mathcal{Q}$-distributor $R^\dagger : \mathcal{Y}^\dagger \nrightarrow \mathcal{X}^\dagger$.

Corollary

The dagger involution in $\mathcal{Q}$ induces a quantaloid isomorphism $(-)^\dagger : \mathbf{Dist}_\mathcal{Q} \to \mathbf{Dist}_\mathcal{Q}^{op}$ ($\mathcal{X} \mapsto \mathcal{X}^\dagger$, $R \mapsto R^\dagger$).

Defining the converse

Goal

Define a converse type of operation $\circlearrowleft : \mathbf{Dist}_{\mathcal{Q}} \to \mathbf{Dist}_{\mathcal{Q}}^{op}$ that is the identity on objects.

Definition

$\mathbf{Matr}_{\mathcal{Q}}$ is the quantaloid where:

- Objects are sets,
- For any two sets X and Y, $\mathbf{Matr}_{\mathcal{Q}}(X, Y)$ is the complete lattice of matrices $\Phi : X \nrightarrow Y$, functions of type $\Phi : X \times Y \to \mathcal{Q}$.

Proposition

The function $| - | : \mathbf{Dist}_{\mathcal{Q}} \to \mathbf{Matr}_{\mathcal{Q}}$ that:

- Maps $\mathcal{X}$ to its underlying set $|\mathcal{X}|$
- Maps $R : \mathcal{X} \nrightarrow \mathcal{Y}$ to the underlying matrix $|R| : |\mathcal{X}| \nrightarrow |\mathcal{Y}|$

is a lax quantaloid morphism.

Defining the converse

Lemma

For any two $\mathcal{Q}$-categories $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{array}{ccc} & \overset{(-)^{\uparrow}}{\longleftarrow} & \\ & \perp & \\ \mathbf{Dist}_{\mathcal{Q}}(\mathcal{X},\mathcal{Y}) & \xrightarrow{\quad |-| \quad} & \mathbf{Matr}_{\mathcal{Q}}(|\mathcal{X}|,|\mathcal{Y}|) \\ & \perp & \\ & \underset{(-)^{\downarrow}}{\longleftarrow} & \end{array}$$

where:

- $\Phi^{\uparrow} := 1_{\mathcal{X}} \bullet \Phi \bullet 1_{\mathcal{Y}}$,
- $\Phi^{\downarrow} := 1_{\mathcal{X}} \blacktriangleright \Phi \blacktriangleleft 1_{\mathcal{Y}}$

Defining the converse

Since for any $\mathcal{Q}$-categories $\mathcal{X}$, $|\mathcal{X}| = |\mathcal{X}^\dagger|$ and $|\mathcal{Y}| = |\mathcal{Y}^\dagger|$:

$$
\begin{array}{ccccc}
\mathbf{Dist}^{op}_{\mathcal{Q}}(\mathcal{X}^\dagger, \mathcal{Y}^\dagger) & \underset{(-)^\downarrow}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Matr}^{op}_{\mathcal{Q}}(|\mathcal{X}^\dagger|, |\mathcal{Y}^\dagger|) & \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}} & \mathbf{Matr}^{op}_{\mathcal{Q}}(|\mathcal{X}|, |\mathcal{Y}|) \\
\uparrow \simeq \downarrow & & & & |-| \uparrow \vdash \downarrow (-)^\uparrow \\
\mathbf{Dist}_{\mathcal{Q}}(\mathcal{X}, \mathcal{Y}) & & \underset{\circlearrowleft}{\overset{\circlearrowright}{\underset{\dashleftarrow}{\overset{\dashrightarrow}{\bot}}}} & & \mathbf{Dist}^{op}_{\mathcal{Q}}(\mathcal{X}, \mathcal{Y})
\end{array}
$$

Definition

For any $\mathcal{Q}$-distributor $R : \mathcal{X} \looparrowright \mathcal{Y}$ let:

- $\circlearrowright R := 1_{\mathcal{Y}} \bullet |R^\dagger| \bullet 1_{\mathcal{X}}$,
- $\circlearrowleft R := 1_{\mathcal{Y}} \blacktriangleright |R|^\dagger \blacktriangleleft 1_{\mathcal{X}}$.

Defining the converse

Theorem

The map $\circlearrowleft : \mathbf{Dist}_{\mathcal{Q}} \to \mathbf{Dist}_{\mathcal{Q}}^{op}$ *that:*

- *Is the identity on* $\mathcal{Q}$*-categories,*
- *Maps every* $\mathcal{Q}$*-distributor* $R : \mathcal{X} \nrightarrow \mathcal{Y}$ *to* $\circlearrowleft R : \mathcal{Y} \nrightarrow \mathcal{X}$*,*

is a function that satisfies the following properties:

- $\circlearrowleft(\bigvee_i R_i) = \bigvee_i \circlearrowleft R_i$,
- $\circlearrowleft(R \bullet S) \leq \circlearrowleft S \bullet \circlearrowleft R$,
- $1_{\circlearrowleft \mathcal{X}} \leq \circlearrowleft 1_{\mathcal{X}}$,
- $R \leq \circlearrowleft\circlearrowleft R$

Define complement type operations

Theorem (Rosenthal [1])

If $\mathcal{Q}$ is a Girard quantale, then $\mathbf{Dist}_{\mathcal{Q}}$ is a Girard quantaloid.

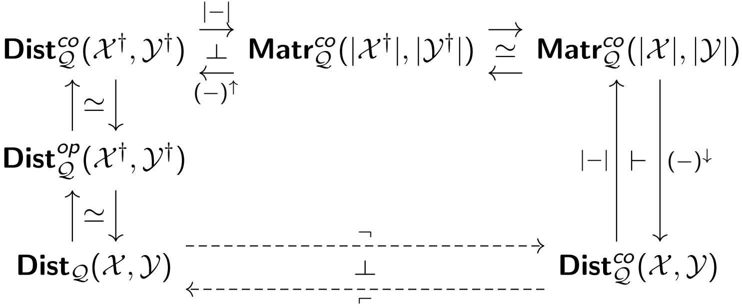
The linear negation $(-)^{\pm} : \mathbf{Dist}_{\mathcal{Q}} \to \mathbf{Dist}_{\mathcal{Q}}^{coop}$ maps every $\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{Y}$ to the $\mathcal{Q}$-distributor $R^{\pm} : \mathcal{Y} \nrightarrow \mathcal{X}$ where $R^{\pm}(y, x) = R(x, y)^{\perp}$.

Corollary

For any two $\mathcal{Q}$-categories $\mathcal{X}$ and $\mathcal{Y}$, there exists a local equivalence betwen $\mathbf{Dist}_{\mathcal{Q}}(\mathcal{X}, \mathcal{Y})$ and $\mathbf{Dist}_{\mathcal{Q}}^{coop}(\mathcal{X}, \mathcal{Y})$.

Define complement type operations

We can define the local adjunction:



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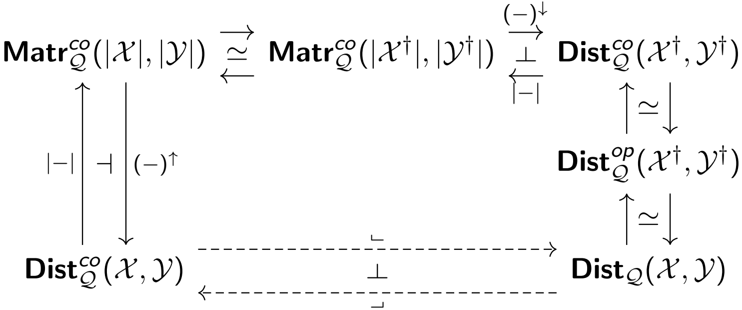

Define complement type operations

Definition

For any $\mathcal{Q}$-co-copresheaf $\varphi : * \nrightarrow \mathcal{X}$, let:

- $\llcorner\varphi := |\varphi|^{\pm\dagger} \bullet X(-,-)$,
- $\ulcorner\varphi := |\varphi|^{\pm\dagger} \blacktriangleleft X(-,-)$
- $\lrcorner\varphi := |\varphi^{\dagger\pm}| \bullet X(-,-)$,
- $\neg\varphi := |\varphi^{\dagger\pm}| \blacktriangleleft X(-,-)$

Lemma

For any $\mathcal{Q}$-category $\mathcal{X}$, the complement-type operations form the adjunctions of $\mathcal{Q}$-categories:

$$\mathcal{U}\mathcal{X} \underset{\overset{\longleftarrow}{\llcorner}}{\overset{\overset{\lrcorner}{\longrightarrow}}{\bot}} (\mathcal{U}\mathcal{X})^{\dagger} \underset{\overset{\longleftarrow}{\neg}}{\overset{\overset{\ulcorner}{\longrightarrow}}{\bot}} \mathcal{U}\mathcal{X}$$

Correspondence between dilations and erosions

Theorem

For any $\mathcal{Q}$-distributor $R : \mathcal{X} \looparrowright \mathcal{Y}$, the following diagrams commute:

$$\begin{array}{ccc} \mathcal{U}\mathcal{Y} & \xrightarrow{-\blacktriangleleft R} & \mathcal{U}\mathcal{X} \\ {\scriptstyle \neg}\big\uparrow & & \big\downarrow{\scriptstyle \lrcorner} \\ (\mathcal{U}\mathcal{Y})^{\dagger} & \xrightarrow[-\bullet \circlearrowleft R]{} & (\mathcal{U}\mathcal{X})^{\dagger} \end{array} \qquad \begin{array}{ccc} \mathcal{U}\mathcal{X} & \xrightarrow{-\bullet R} & \mathcal{U}\mathcal{Y} \\ {\scriptstyle \llcorner}\big\uparrow & & \big\downarrow{\scriptstyle \neg} \\ (\mathcal{U}\mathcal{X})^{\dagger} & \xrightarrow[-\blacktriangleleft \circlearrowleft R]{} & (\mathcal{U}\mathcal{Y})^{\dagger} \end{array}$$

Conclusion and Future Work

- Framework for Colour/Greyscale Graph MM within Quantale Enriched Category Theory

Concept	$\mathcal{Q}$-enriched generalisation
Grid	$\mathcal{Q}$-category $\mathcal{X}$
Relation	$\mathcal{Q}$-distributor $R : \mathcal{X} \nrightarrow \mathcal{X}$
Image Space	$\mathcal{Q}$-category $\mathcal{U}\mathcal{X}$
Dilation	$- \bullet R$
Erosion	$- \blacktriangleleft R$
Converse	$\circlearrowleft R : \mathcal{X} \nrightarrow \mathcal{X}$
Complement	$\neg, \lrcorner, \ulcorner, \llcorner$

- Role of the right converse.
- Build a modal logic that allows for spatial reasoning of Greyscale/Colour Graph MM.

Thank You Very Much!

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