A coarse Erdős-Pósa theorem

Jungho Ahn

Durham University

Joint work with

J. Pascal Gollin (University of Primorska)
Tony Huynh (Sapienza University of Rome)
O-joung Kwon (Hanyang University and IBS)

BCTCS 2025

Feedback Vertex Set

Feedback Vertex Set

Input: A graph G and a positive integer k.
Task: Find a set S of at most k vertices such that $G - S$ is a forest.

Feedback Vertex Set

Feedback Vertex Set

Input: A graph G and a positive integer k.
Task: Find a set S of at most k vertices such that $G - S$ is a forest.

Observation

If G has t vertex-disjoint cycles, then we should delete at least t vertices to make the resulting graph a forest.

Erdős-Pósa theorem

Theorem (Erdős and Pósa, 1965)

For every integer $k \geqslant 1$, every graph G has either

- *k vertex-disjoint cycles or*
- *a set X of $\mathcal{O}(k \log k)$ vertices such that $G - X$ has no cycles.*

Erdős-Pósa theorem

Theorem (Erdős and Pósa, 1965)

For every integer $k \geqslant 1$, every graph G has either

- *k vertex-disjoint cycles or*
- *a set X of $\mathcal{O}(k \log k)$ vertices such that $G - X$ has no cycles.*

Question

What if we consider cycles which are not only vertex-disjoint, but also have no edge between them?

Induced packing

Definition (Induced packing)

An induced packing of cycles in a graph G is a collection $\mathcal{C}$ of pairwise vertex-disjoint cycles such that G has no edge between distinct cycles in $\mathcal{C}$.

Induced packing

Definition (Induced packing)

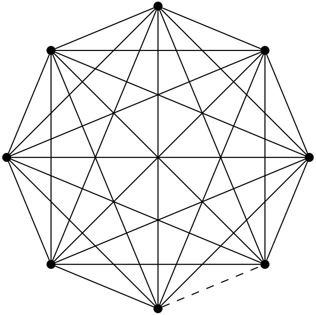
An induced packing of cycles in a graph G is a collection $\mathcal{C}$ of pairwise vertex-disjoint cycles such that G has no edge between distinct cycles in $\mathcal{C}$.

Natural attemt to extend Erdős-Pósa theorem

For every integer $k \geqslant 1$, does every graph G have either

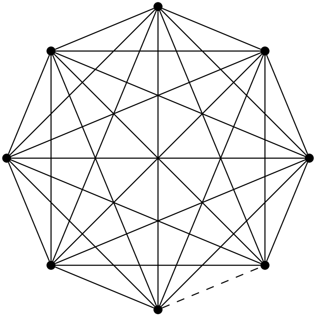
- an induced packing of k cycles or
- a set X of $\mathcal{O}(k \log k)$ vertices such that $G - X$ has no cycles?

Conterexample



Large complete graph has no induced packing of two cycles, and we have to remove almost all vertices to eliminate all cycles.

Conterexample



However, any single vertex dominates all cycles.

Motivation

"Can we either find a large induced packing of cycles, or dominate all cycles by a small set?"

Main result 1

Definition (Ball)

For a graph G, an integer $d \geqslant 0$, and a set $X \subseteq V(G)$, the ball of radius d around X in G, denoted by $B_G(X, d)$, is the set of vertices of G which are at distance at most d from X.

Main result 1

Definition (Ball)

For a graph G, an integer $d \geqslant 0$, and a set $X \subseteq V(G)$, the ball of radius d around X in G, denoted by $B_G(X, d)$, is the set of vertices of G which are at distance at most d from X.

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k \geqslant 1$, every graph G has either

- *an induced packing of k cycles or*
- *a set X of $\mathcal{O}(k \log k)$ vertices such that $G - B_G(X, 1)$ has no cycles.*

Moreover, we can find the induced packing or the set in polynomial time.

Main result 1 for planar graphs

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k \geqslant 1$, every planar multigraph G has either

- *an induced packing of k cycles or*
- *a set X of at most $5k$ vertices such that $G - B_G(X, 1)$ has no cycles.*

Moreover, we can find the induced packing or the set in polynomial time.

Main result 2

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $d \geqslant 1$, every graph G has either

- *two cycles at distance more than d or*
- *a set X of vertices such that $|X| \leqslant 12$ and $G - B_G(X, 3d)$ has no cycles.*

Moreover, we can find the packing or the set in polynomial time.

Proof outline for the main result 1

- **Step 1:** Construct a subgraph $\mathcal{H}$ where every vertex has degree between 2 and 4.

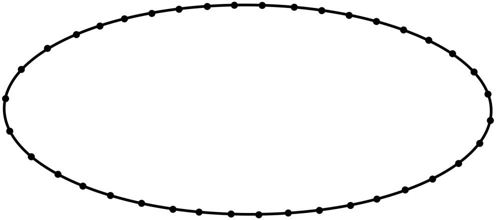
Proof outline for the main result 1

- **Step 1:** Construct a subgraph $\mathcal{H}$ where every vertex has degree between 2 and 4.
- **Step 2:** Find one of the following:
  - a small set of vertices in $\mathcal{H}$ which dominates all cycles in G or
  - a large induced packing of cycles in $\mathcal{H}$.

Proof outline for the main result 1

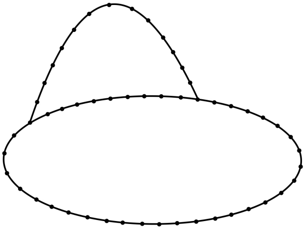
- **Step 1:** Construct a subgraph $\mathcal{H}$ where every vertex has degree between 2 and 4.
- **Step 2:** Find one of the following:
  - a small set of vertices in $\mathcal{H}$ which dominates all cycles in G or
  - a large induced packing of cycles in $\mathcal{H}$.
- **Step 3:** Show that the packing is indeed an induced packing in G when the girth of G is at least 5.

Step 1: Construct the subgraph $\mathcal{H}$



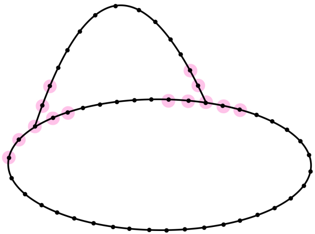
Find a shortest cycle.

Step 1: Construct the subgraph $\mathcal{H}$



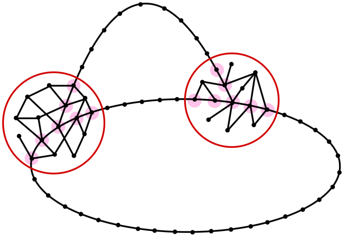
Add a shortest ear.

Step 1: Construct the subgraph $\mathcal{H}$



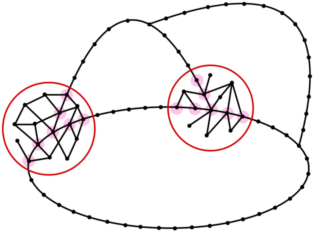
Take a ball of radius 2 in the current $\mathcal{H}$ around the branch vertices.

Step 1: Construct the subgraph $\mathcal{H}$



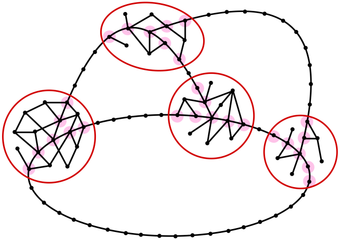
Take a ball of radius 1 in G around the chosen vertices.

Step 1: Construct the subgraph $\mathcal{H}$



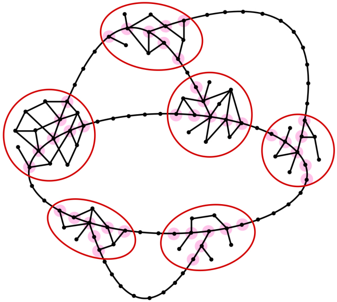
Recursively add a shortest ear avoiding the vertices in the red circles.

Step 1: Construct the subgraph $\mathcal{H}$



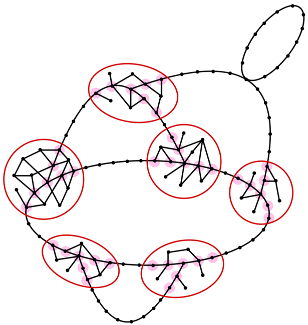
Recursively add a shortest ear avoiding the vertices in the red circles.

Step 1: Construct the subgraph $\mathcal{H}$



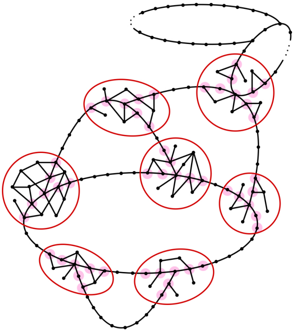
Recursively add a shortest ear avoiding the vertices in the red circles.

Step 1: Construct the subgraph $\mathcal{H}$



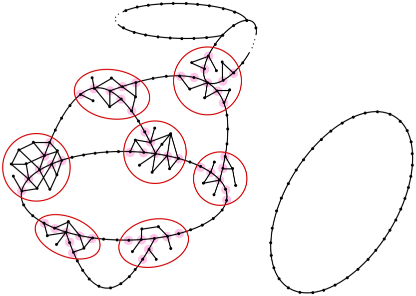
If we're stuck, then add a shortest cycle intersecting the current $\mathcal{H}$ only at one vertex.

Step 1: Construct the subgraph $\mathcal{H}$



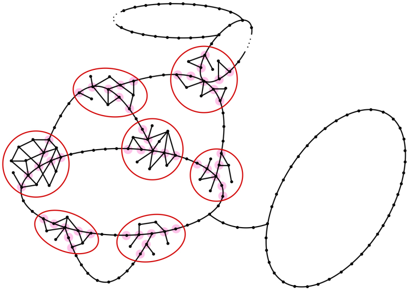
Keep the process of adding shortest ears as before.

Step 1: Construct the subgraph $\mathcal{H}$



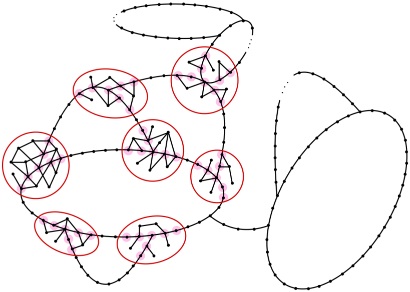
If we're stuck again, then add a shortest cycle outside.

Step 1: Construct the subgraph $\mathcal{H}$



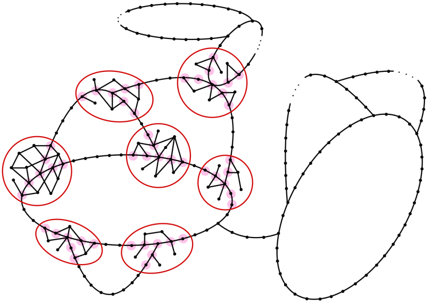
Keep the process of adding shortest ears as before.

Step 1: Construct the subgraph $\mathcal{H}$



Keep the process of adding shortest ears as before.

Step 1: Construct the subgraph $\mathcal{H}$



Let $\mathcal{H}$ be the final graph.

Proof outline

- **Step 1:** Construct a subgraph $\mathcal{H}$ where every vertex has degree between 2 and 4.
- **Step 2:** Find one of the following:
  - a small set of vertices in $\mathcal{H}$ which dominates all cycles in G or
  - a large induced packing of cycles in $\mathcal{H}$.
- **Step 3:** Show that the packing is indeed an induced packing in G when the girth of G is at least 5.

Step 2-1: Small set dominating all cycles

Lemma

Let $\mathcal{H}$ be a maximal coarse ear-decomposition in a graph G. Let X be the set obtained from Y_{t,μ_t} by adding one arbitrary vertex from each component of $\mathcal{H}$ which is a cycle. Then $G - B_G(X, 1)$ has no cycle.

Step 2-1: Small set dominating all cycles

Lemma

Let $\mathcal{H}$ be a maximal coarse ear-decomposition in a graph G. Let X be the set obtained from Y_{t,μ_t} by adding one arbitrary vertex from each component of $\mathcal{H}$ which is a cycle. Then $G - B_G(X, 1)$ has no cycle.

$\Rightarrow$ The purple vertices (+ some extra vertices) dominate all cycles in G.

Step 2-1: Small set dominating all cycles

Lemma

Let $\mathcal{H}$ be a maximal coarse ear-decomposition in a graph G. Let X be the set obtained from Y_{t,μ_t} by adding one arbitrary vertex from each component of $\mathcal{H}$ which is a cycle. Then $G - B_G(X, 1)$ has no cycle.

$\Rightarrow$ The purple vertices (+ some extra vertices) dominate all cycles in G.

$\Rightarrow$ If $\mathcal{H}$ has a few branch vertices, then we are done. Thus, we may assume that $\mathcal{H}$ has many branch vertices.

Step 2-2: Large induced packing in $\mathcal{H}$

How many branch vertices do we want?

Theorem (Simonovits, 1967)

There is a function $s(k) = \Theta(k \log k)$ such that for every integer $k \geqslant 1$, every cubic multigraph with at least $s(k)$ vertices has k vertex-disjoint cycles.

Step 2-2: Large induced packing in $\mathcal{H}$

How many branch vertices do we want?

Theorem (Simonovits, 1967)

There is a function $s(k) = \Theta(k \log k)$ such that for every integer $k \geqslant 1$, every cubic multigraph with at least $s(k)$ vertices has k vertex-disjoint cycles.

$\Rightarrow$ We assume that $\mathcal{H}$ has at least $s(k) + 30(k-1)$ branch vertices.

Step 2-2: Large induced packing in $\mathcal{H}$

How many branch vertices do we want?

Theorem (Simonovits, 1967)

There is a function $s(k) = \Theta(k \log k)$ such that for every integer $k \geqslant 1$, every cubic multigraph with at least $s(k)$ vertices has k vertex-disjoint cycles.

$\Rightarrow$ We assume that $\mathcal{H}$ has at least $s(k) + 30(k-1)$ branch vertices.

$\Rightarrow$ To find a large induced packing in $\mathcal{H}$, we will first remove at most $30(k-1)$ branch vertices and then apply Simonovits's theorem.

Step 2-2: Large induced packing in $\mathcal{H}$

If a graph G has a cycle C of length at most 4, then we apply induction on $G - B_G(V(C), 1)$. Thus, we now assume that G has girth at least 5.

Step 2-2: Large induced packing in $\mathcal{H}$

If a graph G has a cycle C of length at most 4, then we apply induction on $G - B_G(V(C), 1)$. Thus, we now assume that G has girth at least 5.

Lemma

Let G be a graph having no cycle of length at most 4 and
let $\mathcal{H} = \bigcup_{i\in[t]} \bigcup_{j\in[\mu_i]} P_{i,j}$ be a maximal coarse ear-decomposition in G.

Step 2-2: Large induced packing in $\mathcal{H}$

If a graph G has a cycle C of length at most 4, then we apply induction on $G - B_G(V(C), 1)$. Thus, we now assume that G has girth at least 5.

Lemma

Let G be a graph having no cycle of length at most 4 and
let $\mathcal{H} = \bigcup_{i \in [t]} \bigcup_{j \in [\mu_i]} P_{i,j}$ be a maximal coarse ear-decomposition in G.

Let $\mathcal{I}$ be the set of pairs (i, j) with $i \in [t]$ and $j \in [\mu_i] \backslash \{1\}$ such that the ends of $P_{i,j}$ are adjacent in $H_{i,j-1}$.

Step 2-2: Large induced packing in $\mathcal{H}$

If a graph G has a cycle C of length at most 4, then we apply induction on $G - B_G(V(C), 1)$. Thus, we now assume that G has girth at least 5.

Lemma

Let G be a graph having no cycle of length at most 4 and
let $\mathcal{H} = \bigcup_{i\in[t]} \bigcup_{j\in[\mu_i]} P_{i,j}$ be a maximal coarse ear-decomposition in G.

Let $\mathcal{I}$ be the set of pairs (i, j) with $i \in [t]$ and $j \in [\mu_i]\setminus\{1\}$ such that the ends of $P_{i,j}$ are adjacent in $H_{i,j-1}$.

If $t \geqslant 2k - 1$ or $|\mathcal{I}| \geqslant 2k - 1$, then G has an induced packing of k cycles.

Step 2-2: Large induced packing in $\mathcal{H}$

If a graph G has a cycle C of length at most 4, then we apply induction on $G - B_G(V(C), 1)$. Thus, we now assume that G has girth at least 5.

Lemma

Let G be a graph having no cycle of length at most 4 and let $\mathcal{H} = \bigcup_{i \in [t]} \bigcup_{j \in [\mu_i]} P_{i,j}$ be a maximal coarse ear-decomposition in G.

Let $\mathcal{I}$ be the set of pairs (i, j) with $i \in [t]$ and $j \in [\mu_i] \setminus \{1\}$ such that the ends of $P_{i,j}$ are adjacent in $H_{i,j-1}$.

If $t \geqslant 2k - 1$ or $|\mathcal{I}| \geqslant 2k - 1$, then G has an induced packing of k cycles.

$\Rightarrow$ Thus, we may assume that both t and $|\mathcal{I}|$ are at most $2(k-1)$.

Step 2-2: Large induced packing in $\mathcal{H}$

Remove a few branch vertices

Let $\mathcal{H}'$ be the graph obtained from $\mathcal{H}$ by removing every vertex v such that either

- $\deg_{\mathcal{H}}(v) = 4$, or
- v is one end of $P_{i,2}$ whose length is 1, or
- v is one end of $P_{i,j}$ for some $(i, j) \in \mathcal{I}$,

and then recursively remove degree-1 vertices.

Step 2-2: Large induced packing in $\mathcal{H}$

Remove a few branch vertices

Let $\mathcal{H}'$ be the graph obtained from $\mathcal{H}$ by removing every vertex v such that either

- $\deg_{\mathcal{H}}(v) = 4$, or
- v is one end of $P_{i,2}$ whose length is 1, or
- v is one end of $P_{i,j}$ for some $(i,j) \in \mathcal{I}$,

and then recursively remove degree-1 vertices.

Note that $\mathcal{H}'$ has at least $s(k)$ branch vertices, so it has a packing $\mathcal{C}$ of k cycles by Simonovits's theorem.

Step 2-2: Large induced packing in $\mathcal{H}$

Why we removed those vertices?

Lemma

Any set of pairwise vertex-disjoint cycles of $\mathcal{H}'$ is an induced packing in $\mathcal{H}$.

$\Rightarrow$ Thus, $\mathcal{C}$ is an induced packing in $\mathcal{H}$.

Proof outline

- **Step 1:** Construct a subgraph $\mathcal{H}$ where every vertex has degree between 2 and 4.
- **Step 2:** Find one of the following:
  - a small set of vertices in $\mathcal{H}$ which dominates all cycles in G or
  - a large induced packing of cycles in $\mathcal{H}$.
- **Step 3:** Show that the packing is indeed an induced packing in G when the girth of G is at least 5.

Step 3: Show that $\mathcal{C}$ is an induced packing

If $\mathcal{C}$ is not an induced packing in G, then $E(G) \setminus E(\mathcal{H})$ has an edge between distinct cycles in $\mathcal{H}$.

Step 3: Show that $\mathcal{C}$ is an induced packing

If $\mathcal{C}$ is not an induced packing in G, then $E(G) \setminus E(\mathcal{H})$ has an edge between distinct cycles in $\mathcal{H}$.

Proposition

If G has girth at least 5, then $\mathcal{H}$ is an induced subgraph of G.

$\Rightarrow$ Thus, $\mathcal{C}$ is also an induced packing in G.

Summary

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k \geqslant 1$, every graph G has either

- *an induced packing of k cycles or*
- *a set X of $\mathcal{O}(k \log k)$ vertices such that $G - B_G(X, 1)$ has no cycles.*

Moreover, we can find the induced packing or the set in polynomial time.

Summary

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k \geqslant 1$, every graph G has either

- *an induced packing of k cycles or*
- *a set X of $\mathcal{O}(k \log k)$ vertices such that $G - B_G(X, 1)$ has no cycles.*

Moreover, we can find the induced packing or the set in polynomial time.

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k \geqslant 1$, every planar multigraph G has either

- *an induced packing of k cycles or*
- *a set X of at most $5k$ vertices such that $G - B_G(X, 1)$ has no cycles.*

Moreover, we can find the induced packing or the set in polynomial time.

Summary

Theorem (A., Gollin, Huynh, and Kwon, 2024)

For every integer $d \geqslant 1$, every graph G has either

- *two cycles at distance more than d or*
- *a set X of vertices such that $|X| \leqslant 12$ and $G - B_G(X, 3d)$ has no cycles.*

Moreover, we can find the packing or the set in polynomial time.

Conjecture

Conjecture (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k, d \geqslant 1$, every graph G has either

- a distance-d packing of k cycles or
- a set X of $O(k \log k)$ vertices such that $G - B_G(X, c \cdot d)$ has no cycles for some constant $c > 0$.

Conjecture

Conjecture (A., Gollin, Huynh, and Kwon, 2024)

For every integer $k, d \geqslant 1$, every graph G has either

- a distance-d packing of k cycles or
- a set X of $O(k \log k)$ vertices such that $G - B_G(X, c \cdot d)$ has no cycles for some constant $c > 0$.

Theorem (Dujmović, Joret, Micek, and Morin, 2024)

For every integer $k, d \geqslant 1$, every graph G has either

- *a distance-d packing of k cycles or*
- *a set X of $O(k^{18} \log^{18} k)$ vertices such that $G - B_G(X, 19d)$ has no cycles for some constant $c > 0$.*

Thank you for your attention!