Proof Translations for Structurally Different Sequent Calculi of Intuitionistic Modal Logic

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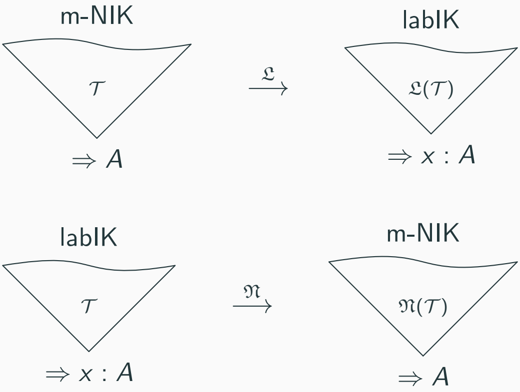
... translate derivations from one calculus into another one.



The Intuitionistic Modal Logic IK

Language of IK

We define the language $\mathcal{L}^{\Box\Diamond}$ by a countably infinite set of propositional atoms $\Phi = \{p, q, r, ...\}$ and some independent logical connectives and modalities.

$$A ::= \bot \mid p \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid \Box A \mid \Diamond A$$

We further include the following abbreviations: $\neg A := (A \to \bot)$ and $\top := (\bot \to \bot)$.

Axiomatisation of IK

The logic IK (Intuitionistic K) was first introduced by Fischer Servi 1984 and popularised by Simpson 1994 who also introduced this axiomatisation.

<u>Axiom schemas</u>

Axioms of IPL

$k_1 : \Box(A \to B) \to (\Box A \to \Box B)$

$k_2 : \Box(A \to B) \to (\Diamond A \to \Diamond B)$

$k_3 : \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

$k_4 : (\Diamond A \to \Box B) \to \Box(A \to B)$

$k_5 : \neg\Diamond\bot$

<u>Rules</u>

$$(\mathrm{mp})\ \frac{A \to B \qquad A}{B}$$

$$(\mathrm{nec})\ \frac{A}{\Box A}$$

IK as a Fragment of Intuitionistic First-order Logic

We can translate IK into intuitionistic first-order logic (IL) via the standard translation.

$$\mathsf{IK} \vdash A \;\Leftrightarrow\; \mathsf{IL} \vdash \forall x ST^x(A)$$

$ST^x(\bot) = \bot$
$ST^x(p) = P(x)$
$ST^x(A \to B) = ST^x(A) \to ST^x(B)$
$ST^x(A \vee B) = ST^x(A) \vee ST^x(B)$
$ST^x(A \wedge B) = ST^x(A) \wedge ST^x(B)$
$ST^x(\Box A) = \forall y(xRy \to ST^y(A))$
$ST^x(\Diamond A) = \exists y(xRy \wedge ST^y(A))$

This also yields some desirable properties of IK:

1. Conservative over IPL
2. Disjunction property
3. $\Box$ and $\Diamond$ are not interdefinable
4. Adding LEM yields the logic K

Semantics for IK

A birelational model is a tuple $\langle W, \leq, R, V \rangle$ with a non-empty set W, a pre-order $\leq \subseteq W \times W$, a relation $R \subseteq W \times W$ and a valuation function $V : \Phi \to \mathcal{P}(W)$. It further satisfies monotonicity ($V(p)$ is upwards closed wrt. $\leq$) and forward/backward confluence.

$$
\begin{tikzcd}
z' \arrow[r, dotted, "R"] & z \\
x \arrow[u, dotted, "\leq"] \arrow[r, "R"'] & y \arrow[u, "\leq"']
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[r, dotted, "R"] & z' \\
x \arrow[u, "\leq"] \arrow[r, "R"'] & y \arrow[u, dotted, "\leq"']
\end{tikzcd}
$$

Truth is defined in the "usual" ways for modal and intuitionistic logic, except for:

$\mathcal{M}, w \Vdash \Box A$ iff for all $v, u \in W$: if $w \leq v$ and vRu then $\mathcal{M}, u \Vdash A$

Proof Theory of IK

Extending Sequents

We work with sequents $\Gamma \Rightarrow \Delta$ (Γ, Δ being finite multisets of formulas), as it is common practice in structural proof theory.

There are mainly two approaches when it comes to defining sequent calculi for modal logics; extending the *structure* and extending the *language* in sequents.

Extending Sequents

We work with sequents $\Gamma \Rightarrow \Delta$ (Γ, Δ being finite multisets of formulas), as it is common practice in structural proof theory.

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Nested sequents: $\Gamma \Rightarrow \Delta, [\Gamma_1 \Rightarrow \Delta_1], ..., [\Gamma_n \Rightarrow \Delta_n]$
where $\Delta_1, ..., \Delta_n$ can also contain nestings again.

Labelled sequents: $\mathcal{R}; \Gamma \Rightarrow \Delta$
uses labelled formulas $x : A$ and xRy, where $\mathcal{R}$ contains only formulas of the form xRy and Γ, Δ only $x : A$ formulas.

A Fully Labelled Calculus

Marin, Morales, and Straßburger 2021 introduced an extension of classical sequent calculus that internalises the full semantics of IK. Some of the rules of labIK are as follows.

$$\mathrm{Id}\ \frac{}{\mathcal{R}, x \leq y; x : p, \Gamma \Rightarrow \Delta, y : p}$$

$$\mathit{Trans}\ \frac{\mathcal{R}, x \leq y, y \leq z, x \leq z; \Gamma \Rightarrow \Delta}{\mathcal{R}, x \leq y, y \leq z; \Gamma \Rightarrow \Delta}$$

$$\Diamond\mathrm{R}\ \frac{\mathcal{R}, xRy; \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, xRy; \Gamma \Rightarrow \Delta, x : \Diamond A}$$

$$\Box\mathrm{R}\ \frac{\mathcal{R}, x \leq y, yRz; \Gamma \Rightarrow \Delta, z : A}{\mathcal{R}; \Gamma \Rightarrow \Delta, x : \Box A}\ (y, z \text{ fresh})$$

A Fully Labelled Calculus

labIK admits the usual structural rules (weakening, label substitution, monotonicity, contraction, and cut).

It is also *fully invertible* (all necessary information stays in the sequent)! This makes backtracking unnecessary, but also makes it carry a lot of information.

A Maehara-Style Nested Calculus

Kuznets and Straßburger 2019 introduced a nested extension of the intuitionistic Maehara calculus.
Some of the rules of m-NIK are as follows.

$$\mathsf{Id}\ \frac{}{\Gamma\{p \Rightarrow p\}} \qquad \Diamond\mathsf{R}\ \frac{\Gamma\{\Rightarrow \Diamond A, [\Sigma \Rightarrow \Pi, A]\}}{\Gamma\{\Rightarrow \Diamond A, [\Sigma \Rightarrow \Pi]\}} \qquad \Box\mathsf{R}\ \frac{\Gamma^{\downarrow}\{\Rightarrow [\Rightarrow A]\}}{\Gamma\{\Rightarrow \Box A\}}$$

The system m-NIK admits the usual structural rules (weakening, contraction, cut).

Unlike labIK, it is not fully invertible as potentially necessary information can get lost when applying $\to$R or $\Box$R (output formulas get deleted).
At the same time, proofs in m-NIK carry much less information.

Proof Translations

Previous Results

Goré and Ramanayake 2014 introduced translations between (simple) tree-labelled and (simple) nested sequents. These formalisms can therefore be considered notational variants.

$\Box\neg p \Rightarrow, [\Rightarrow p \vee q, [\Rightarrow \Diamond p]]$ $\qquad$ $xRy, yRz; x : \Box\neg p \Rightarrow y : p \vee q, z : \Diamond p$

$$\begin{array}{ccc}
\Box\neg p \Rightarrow & \qquad & x : \Box\neg p \Rightarrow \\
\big\downarrow & & \big\downarrow R \\
\Rightarrow p \vee q & & \Rightarrow y : p \vee q \\
\big\downarrow & & \big\downarrow R \\
\Rightarrow \Diamond p & & \Rightarrow z : \Diamond p
\end{array}$$

This allowed them to find effective translation between derivations and also compare systems of different formulations.

Translating from m-NIK to labIK

Translating from m-NIK to labIK

The main idea for our work is to translate each rule of m-NIK separately into a derivation in labIK (including potentially some admissible rules).

For example:

$$\vee\mathsf{R}\ \frac{\Rightarrow \Box p, [\Rightarrow p, q]}{\Rightarrow \Box p, [\Rightarrow p \vee q]} \quad \rightsquigarrow \quad \frac{xRy; \Rightarrow x : \Box p, y : p, y : q}{xRy; \Rightarrow x : \Box p, y : p \vee q}\ \vee\mathsf{R}$$

In both cases:

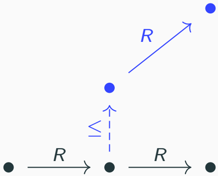
$$\begin{tikzcd} \Rightarrow \Box p \arrow[r, dash] & {} \arrow[r] \arrow[d, dotted] & \Rightarrow p, q \\ \Rightarrow \Box p \arrow[r, dash] & {} \arrow[r] & \Rightarrow p \vee q \end{tikzcd}$$

Translating from m-NIK to labIK

But what about rules that introduce $\leq$-formulas?

Answer: Make a macro rule, in which we **Lift the sequent**.

$$\Box\mathrm{R}\ \frac{\mathcal{R}, x \leq y, yRz; \Gamma \Rightarrow \Delta, z : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A}\ (y, z \text{ fresh})$$



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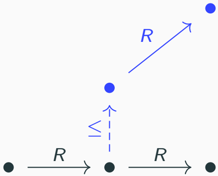
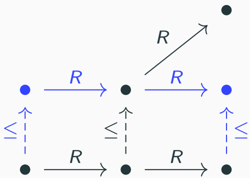
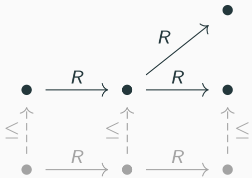


$\mathrm{F}_1\ \mathrm{F}_2\ \mathrm{mon}_L$ $\rightsquigarrow$



wk $\rightsquigarrow$



Translating from m-NIK to labIK

Theorem 1

For any nested sequent Γ, if m-NIK $\vdash \Gamma$ then labIK $\vdash \mathfrak{L}^x(\Gamma)$ (where $\mathfrak{L}^x(\Gamma)$ is the labelled form of Γ). Furthermore, the translation is effective.

Translating from labIK to m-NIK

Translating from labIK to m-NIK

Initial observation: Not all proof trees from labIK are translatable into m-NIK because labIK is fully invertible and m-NIK is not.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x \leq y, x \leq z, zRu, u \leq u, x : \Box(p \wedge q), u : p, u : q, y : q \Rightarrow u : p, y : p}\,\text{Ax}}{x \leq y, x \leq z, zRu, x : \Box(p \wedge q), u : p, u : q, y : q \Rightarrow u : p, y : p}\,\mathit{Ref}}{x \leq y, x \leq z, zRu, x : \Box(p \wedge q), u : p \wedge q, y : q \Rightarrow u : p, y : p}\,\wedge\text{L}}{x \leq y, x \leq z, zRu, x : \Box(p \wedge q), y : q \Rightarrow u : p, y : p}\,\Box\text{L}}{x \leq y, x : \Box(p \wedge q), y : q \Rightarrow x : \Box p, y : p}\,\Box\text{R}}{x : \Box(p \wedge q) \Rightarrow x : \Box p, x : q \to p}\,\to\text{R}$$

$$\begin{tikzcd}
y & z \arrow[r, "R"] & u \\
& x \arrow[ul, dashed, "\leq"] \arrow[u, dashed, "\leq"'] &
\end{tikzcd}$$

Translating from labIK to m-NIK

Before translating we have to ensure that the labIK derivations have the correct form: All sequents in the proof tree must be *linearly layered* (no branching in $\leq$).

This allows one to always find a maximum layer (wrt. $\leq$), which we then can translate into a simple nested sequent.

There are generally two ways to change the proof trees of labIK such that they become "linear": Edit them *directly* or *rebuild* a new derivation under a certain procedure. We did the latter, as it turned out to be much easier.

Searching for Linear Proof Trees

The following lemma allows us to construct linearly layered proof trees.

Lemma (single succession)

Let $\mathcal{R}; \Gamma \Rightarrow \Delta$ be a relationally saturated sequent, then
$\mathsf{labIK} \vdash \mathcal{R}; \Gamma \Rightarrow \Delta$ iff $\mathsf{labIK} \vdash \mathcal{R}; \Gamma \Rightarrow x : C$ for some $x : C \in \Delta$.
We call $x : C$ the single succedent of the sequent.

Linear Search Algorithm

0. Start with a derivable sequent $\Rightarrow x : A$.
1. *Saturate* the leaves of $\mathcal{T}_i$.
2. If all leaves of $\mathcal{T}_i$ are initial sequents, terminate.
   $\rightarrow$ A linear proof of $\Rightarrow x : A$ is obtained.
3. Otherwise, pick a non-axiomatic leaf sequent $\mathfrak{S}'$ in $\mathcal{T}_i$.
   (a) Compute the *lifting* $\mathfrak{S} \otimes \mathfrak{S}{\uparrow}^{x:F}$ (if possible) and go back to Step 1 ($i \mapsto i+1$).
   (b) Otherwise, *backtrack*.

NB: We assume an already derivable formula for our proof search. For actually defining a proper *decision procedure* one also has to incorporate loop checks.

Translating from labIK to m-NIK

Labelled sequents occurring in the algorithm might bare some structure like this

$$
\begin{tikzcd}
\bullet \arrow[r, "R"] & \bullet \arrow[r, "R"] & \bullet \\
\bullet \arrow[u, dashed, "\leq"'] \arrow[r, "R"'] & \bullet \arrow[u, dashed, "\leq"'] \arrow[r, "R"'] & \bullet \arrow[u, dashed, "\leq"'] \\
\bullet \arrow[u, dashed, "\leq"'] & &
\end{tikzcd}
$$

and will be translated into a nested sequent by only considering the top layer.

Translating from m-NIK to labIK

Theorem II

For any formula $A \in \mathcal{L}^{\Box\Diamond}$, if labIK $\vdash \Rightarrow x : A$ then a derivation tree for m-NIK $\vdash \Rightarrow A$ can be effectively obtained.

Corollary

m-NIK and labIK are sound and complete wrt. each other.
Also, for any nested sequent Γ: m-NIK $\vdash \Gamma$ iff labIK $\vdash \mathfrak{L}^x(\Gamma)$.

Corollary

For any formula $A \in \mathcal{L}$:

1. If labG3I $\vdash A$ then a derivation tree of m-G3i $\vdash A$ can be effectively obtained.
2. If m-G3i $\vdash A$ then a derivation tree of labG3I $\vdash A$ can be effectively obtained.

Conclusion

Conclusion

Summary

We introduced (effective) proof translations between the bi-labelled system labIK and the Maehara-style nested system m-NIK.

This establishes direct completeness between these calculi.

The result also reduces to their modal-free counterparts m-G3i and G3I.

Conclusion

Future Works

- finding translations for other modal logics (e.g. modal and intermediate extensions, or constructive modal logics)
- make linearisation of labIK proofs more effective
- build connections to other calculi, or apply the method to other logics
- implementation and complexity analysis

References

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Q&A

Thank you!

Fischer Servi's Axiomatisation

Axiom schemas

Axioms of IPL

$\Box(A \wedge B) \to (\Box A \wedge \Box B)$

$\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$

$\Diamond(A \to B) \to (\Box A \to \Diamond B)$

$(\Diamond A \to \Box B) \to \Box(A \to B)$

$\neg\Diamond\bot$

Rules

$$(\mathrm{mp})\ \frac{A \to B \qquad A}{B}$$

$$(\mathrm{nec}_{\Box})\ \frac{A \to B}{\Box A \to \Box B}$$

$$(\mathrm{nec}_{\Diamond})\ \frac{A \to B}{\Diamond A \to \Diamond B}$$

Example of two Frames