>>> Bounded Henkin quantifiers and the exponential time hierarchy

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BCTCS, Glasgow

>>> Plan of talk

Henkin quantifiers

>>> Plan of talk

Henkin quantifiers + Bounded quantifiers

>>> Plan of talk

Henkin quantifiers + Bounded quantifiers = Bounded Henkin quantifiers

Henkin quantifiers

>>> Hintikka's sentence

''Some relative of each villager and some relative of each townsman hate each other.''

$$(V(x_1) \land T(x_2)) \to (R(x_1, y_1) \land R(x_2, y_2) \land H(y_1, y_2))$$

>>> Hintikka's sentence

''Some relative of each villager and some relative of each townsman hate each other.''

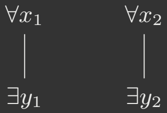
$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} (V(x_1) \wedge T(x_2)) \rightarrow (R(x_1, y_1) \wedge R(x_2, y_2) \wedge H(y_1, y_2))$$

>>> Definition by examples

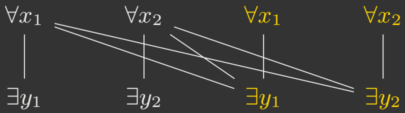
$$H^2_2 = \begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} = \begin{array}{ccc} \forall x_1 & \qquad & \forall x_2 \\ \Big| & & \Big| \\ \exists y_1 & & \exists y_2 \end{array}$$

>>> Definition by examples

$$H_2^2 = \begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix} =$$



$$H_2^2 H_2^2 =$$



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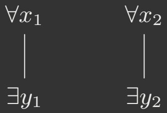

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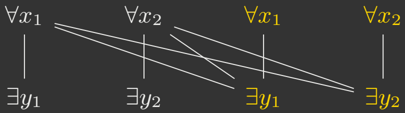


Henkin quantifier

A triple $Q = (A, E, d)$ such that $d \subseteq A \times E$. A Hekin quantifier is called standard if it can be written like a matrix.

>>> Interpreting Henkin quantifiers

Interpret by Skolemisation:

$$\exists f \exists g \forall x_1 \forall x_2 \, (V(x_1) \wedge T(x_2)) \rightarrow (R(x_1, f(x_1)) \wedge R(x_2, g(x_2)) \wedge H(f(x_1), g(x_2))$$

$L(H)$

Language of first-order logic extended by Henkin quantifiers

* $L(H)$ cannot be recursively axiomatised (Erhenfeucht)
* $L(H)$ equivalent to existential second-order logic (Enderton-Walkoe)
* Over finite structures, $L(H)$ can express exactly NP predicates (Blass-Gurevich)

>>> Prenexing and standardisation

Positive formula

An $L(H)$ formula where Henkin quantifiers occur under an even number of negations.

Proposition

Let ϕ be a positive $L(H)$ formula. There exists an H-formula $Q\psi$ such that (i) Q is standard (ii) ψ is a quantifier-free, and (iii) ϕ and $Q\psi$ are equivalent.

Proposition

Every $L(H)$ formula is equivalent to an $L(H)$ formula of the form $R\neg Q_0\neg Q_1\ldots\neg Q_n\phi$ where R is either $\neg Q$ or Q, and $Q, Q_0, \ldots, Q_n$ are standard Henkin quantifiers.

Bounded Quantifiers and Computational Complexity

>>> Is this descriptive complexity?

Descriptive complexity

* Fixed vocabulary and class of formulas $\mathcal{F}$
* A property P is definable if there is a formula $\phi \in \mathcal{F}$ in this syntax such that the set of finite models satisfying ϕ is exactly the set of models with property P
* $\mathcal{F}$ captures a complexity class $\mathcal{C}$ if the properties checkable in $\mathcal{C}$ are exactly the ones definable in $\mathcal{F}$
* Model theoretic approach

Bounded arithmetic

* Fixed vocabulary, class of formulas $\mathcal{F}$, and an infinite model $\mathbb{N}$
* A predicate $R \subseteq \mathbb{N}^2$ is definable if there is a formula $\phi(x, y) \in \mathcal{F}$ in this syntax such that $R = \{(a, b) \mid \mathbb{N} \models \phi(a, b)\}$
* $\mathcal{F}$ captures a complexity class $\mathcal{C}$ if $R \in \mathcal{C}$ iff R is definable in $\mathcal{F}$
* Proof-theoretic approach: want to study theories

>>> Language

Terms

$$t, t' ::= x \in Var \mid 0 \mid S(t) \mid t + t' \mid t \cdot t' \mid |t| \mid t \sharp t' \mid \lfloor t/2 \rfloor$$

where $|t|$ is intended to be interpreted as the number of digits in the binary representation of t and $t \sharp t'$ as $2^{|t||t'|}$.

Formulas

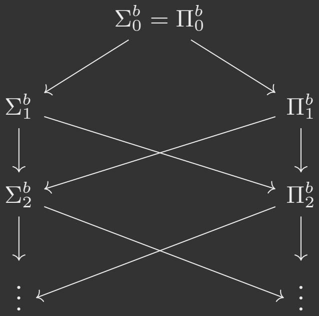
$$\phi, \psi ::= s \leq t \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \exists x \leq s\ \phi \mid \forall x \leq s\ \phi$$

Quantifiers of the form $Qx \leq |s|\ \phi$ are said to be sharply bounded. A formula is sharply bounded if all its quantifiers are sharply bounded.

>>> The Bounded Arithmetic Hierarchy

* $\Sigma_0^b = \Pi_0^b$ are the set of sharply bounded formulas.
* $\Sigma_{i+1}^b = \{\exists x \leq s.\phi \mid \phi \in \Pi_i^b\}$ modulo prenex operations.
* $\Pi_{i+1}^b = \{\forall x \leq s.\phi \mid \phi \in \Sigma_i^b\}$ modulo prenex operations.

[Arbitrary sharply bounded quantifiers allowed in the 2nd and 3rd case]



>>> Capturing complexity classes

Proposition

Predicate $R \subseteq \mathbb{N}^k$ definable by a Σ_0^b formula $\implies$ $R \in P$.

Theorem (Kent-Hodgson'82)

Predicate $R \subseteq \mathbb{N}^k$ definable by a Σ_1^b formula $\iff$ $R \in NP$.

Corollary

Bounded Arithmetic Hierarchy corresponds to PH.

* Starting point of uniform proof complexity
* Consider weak sub-theories of PA in this language
* Characterise complexity classes in the sense that a function is provably total in a theory iff it belongs to a given complexity class.

>>> Second-order Bounded Arithmetic

* Language with second-order bounded quantification
* Captures the exponential hierarchy
* In particular, $\Sigma_1^{1,b}(\mathbb{N}) = NEXP$

Bounded Henkin Quantifiers

>>> Bounded Henkin quantifiers

Pretty much does what it says on the tin:

$$\begin{pmatrix} \forall x_1 \exists y_1 \\ \forall x_2 \exists y_2 \end{pmatrix}$$

Pretty much does what it says on the tin:

$$\begin{pmatrix} \forall x_1 \leq s_1 \; \exists y_1 \leq t_1 \\ \forall x_2 \leq s_2 \; \exists y_2 \leq t_2 \end{pmatrix}$$

H-formulas

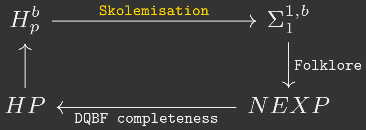
Formulas in the language of bounded arithmetic with bounded Henkin quantifiers.

Main result

Predicate $R \in \mathbb{N}^k$ definable by a positive H-formula $\iff$ $R \in NEXP$

>>> Proof technique

$H_p^b := \{R \subseteq \mathbb{N}^k \mid R$ definable by a positive H-formula$\}$



Few points about Skolemisation

* There is polynomial bounded Gödel encoding of pairs:

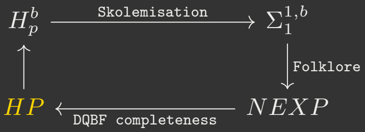
$$\ulcorner \langle m, n \rangle \urcorner := \cancel{2^m 3^n} \left\lfloor \frac{m+n}{2} \right\rfloor m + n$$

* Bounded arithmetic can be bootstrapped with pairing function β.

$$\beta(i, \ulcorner \langle a_1, \ldots, a_k \rangle \urcorner) = \begin{cases} n & \text{if } i = 0; \\ a_i & \text{if } 1 \leq i \leq k \end{cases}$$

* Therefore, Skolem functions can be replaced by polynomially bounded predicates.

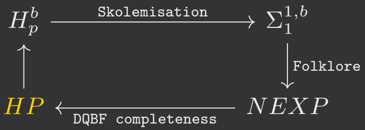
>>> HP sauce



$NP :=$ set of languages L such that there exists a polynomial p and a poly time TM M such that

$$x \in L \iff \exists u \leq p(|x|)\ M(x, u) = 1.$$

>>> HP sauce



$NP :=$ set of languages L such that there exists a polynomial p and a poly time TM M such that

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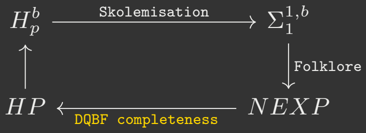
$H_2^2P :=$ set of languages L such that there exists polynomials p_1, q_1, p_2, q_2 and a poly time TM M such that

$$x \in L \iff \begin{pmatrix} \forall x_1 \leq p_1(|x|) & \exists y_1 \leq q_1(|x|) \\ \forall x_2 \leq p_2(|x|) & \exists y_2 \leq q_2(|x|) \end{pmatrix} M(x, x_1, y_1, x_2, y_2) = 1.$$

$HP := \bigcup_Q QP$ [Q ranging over Henkin quantifiers]
Trivially, $HP \subseteq H_p^b$.

>>> A NEXP-complete problem



DQBF

A formula of the form $Q\psi$ where Q is a Henkin quantifier and ψ is a quantifier-free Boolean formula

Theorem (Peterson-Reif'79)

DQBF satisfiability is $NEXP$-complete.

Clearly, DQBF satisfiability $\in HP$.

Further Results

$H := \{R \subseteq \mathbb{N}^k \mid R$ definable by H-formula$\}$

Proposition

$H \subseteq \Delta_2^{1,b}$

Proof Sketch.

Let R be defined by a H-formula ϕ. Then, $\phi = P\neg Q_1 \ldots \neg Q_n \phi$ where P is either $\neg Q$ or Q, and ψ quantifier-free. Induct on n.

* Base case: previous result
* Induction case: use an encoding of the axiom of choice and the following identity:

$$\exists f \forall x \forall g \exists y \phi(x, y, f(x), g(y)) \equiv \forall g \exists f \forall x \exists y \phi(x, y, f(x), g(x, f(x), y))$$

>>> Construing H as a complexity class

GDQBF

A formula of the form $R\neg Q_0 \ldots \neg Q_n \phi$ where $R = Q$ or $R = \neg Q$, $Q_0, \ldots, Q_n$ are Henkin quantifiers, and ψ is a quantifier-free Boolean formula.

Theorem

GDQBF satisfiability is H-complete.

>>> Curtain call

Conclusion

* Defined bounded Henkin quantifiers in the language of bounded arithmetic
* Positive formulas exactly capture $NEXP$
* Arbitrary formulas not much more expressive: collapses at Δ_2 of the exponential hierarchy

Future work

* Descriptive complexity conjecture: $H \subsetneq \Delta_2^{EXP}$ (modulo some complexity theory assumptions)
* Bounded arithmetic: consider theories with induction on positive H-formulas. Can we formalise our result in this theory?
* Proof complexity: connections between (D)QBF solving algorithms and such theories...

See ICLA 2025 paper for more details

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Thanks!