UNIVERSITY OF LIVERPOOL

LEVERHULME TRUST

The Leverhulme Research Centre for Functional Materials Design

Fast and Safe Scheduling of Robots

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BCTCS 2025

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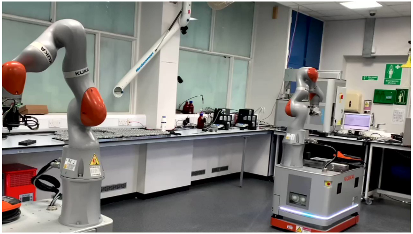
Contents

Motivation



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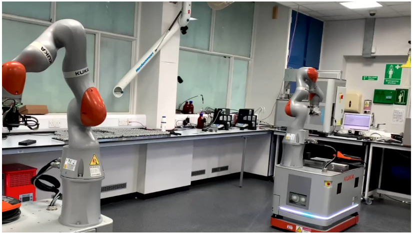

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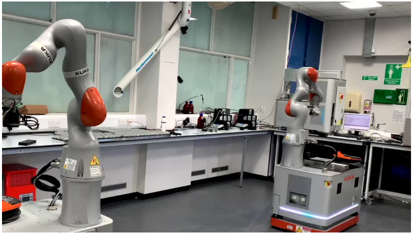

- Many labs around the world have mobile robots to automate chemistry experiments.
- We are interested in routing these robots in order to complete tasks based around the lab as fast as possible.
- Our model can also be relevant to manufacturing and logistics.

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The Setting:

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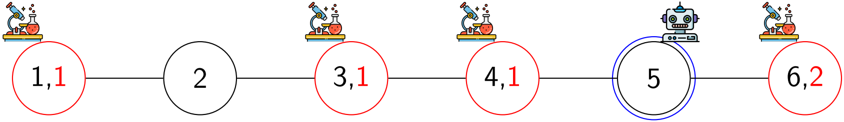
[5] Total number of time steps taken by all schedules in the set

Contents

One robot on a path



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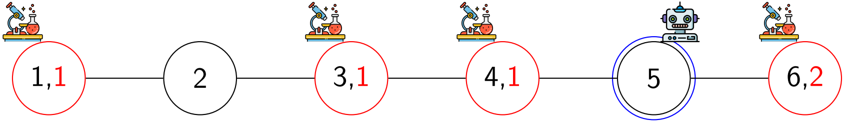

- $C_1(P, T, sv) := \min(|sv - v_{t_1}|, |sv - v_{t_m}|) + v_{t_m} - v_{t_1} + \sum_{i \in [m]} d_i.$

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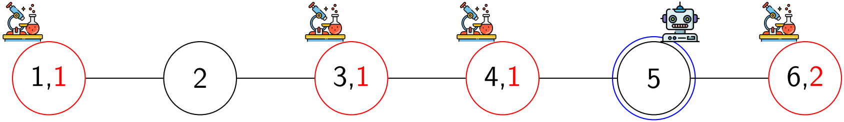

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- For the example above:

$$C_1(P_6, \{(1,1), (3,1), (4,1), (6,2)\}, 5) = min(5-1, 6-5) + 6 - 1 + 5 = 11$$

2-Partition Algorithm

- For two robots, R_L and R_R, starting on vertices sv_L and sv_R, we wish to split the tasks into two where R_L completes $t_1, ..., t_q$ and R_R does $t_{q+1}, ..., t_m$.

2-Partition Algorithm

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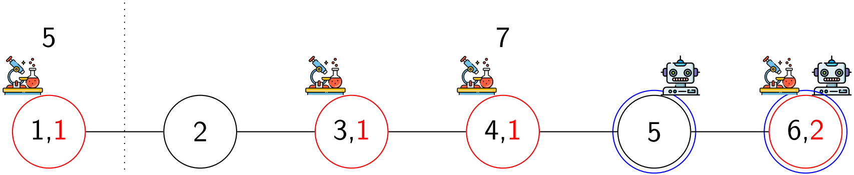
- We determine the value of q by finding the value which minimises

$$\max(C_1(P_{\max(sv_L, i_q)}, (t_1, , \ldots, t_q), sv_L), C_1(P_{1,\min(i_{q+1}, sv_R), m}, (t_{q+1}, \ldots, t_m), sv_R))$$

.

Partition Example



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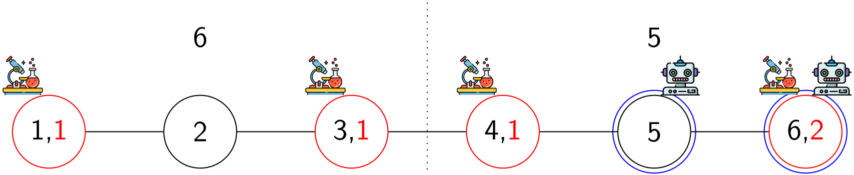

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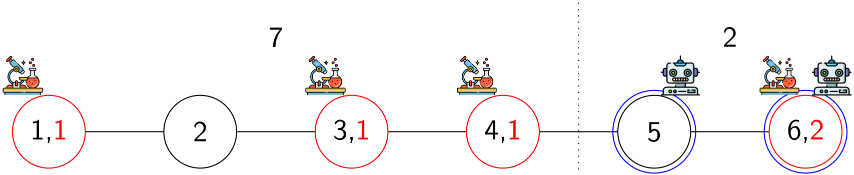

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Sketch of the $O(kmn)$ dynamic programming algorithm:

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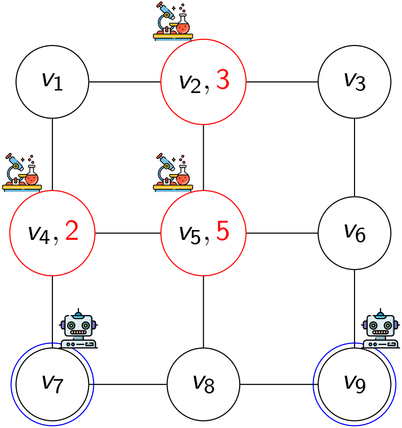
- When $k = 2$ this is the same as the 2-Partition Algorithm.

Contents

Grid graphs

A naïve way of approximating $k-$ Robot Scheduling on a grid is by finding an Hamiltonian Path and executing the partition algorithm on that path.



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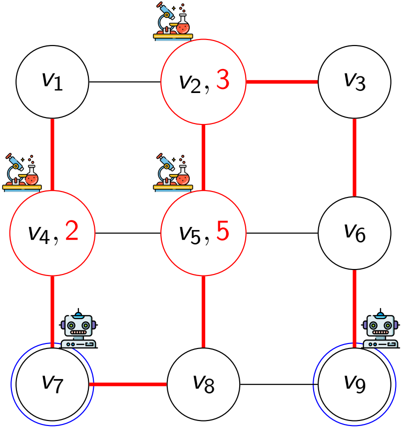

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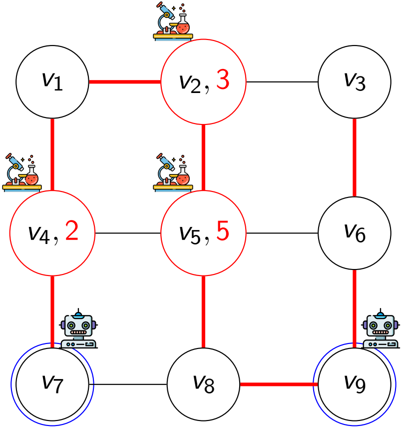

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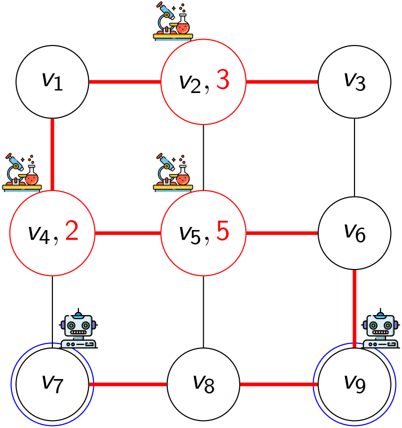

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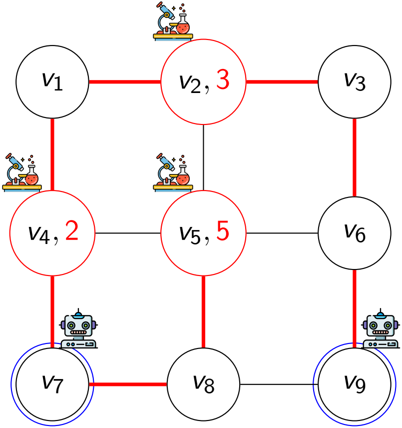

Algorithm for Cycles

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Lemma

Given an instance of k-ROBOT SCHEDULING *on a cycle* $G = (V, E)$ *with a set of equal duration tasks* $T = \{t_1, \ldots, t_m\}$ *and robots* $r_1, \ldots, r_k$*, there exists a fastest collision-free task-completing schedule* C *such that there exists some edge* $e \in E$ *that is not traversed by any robot in* C*.*

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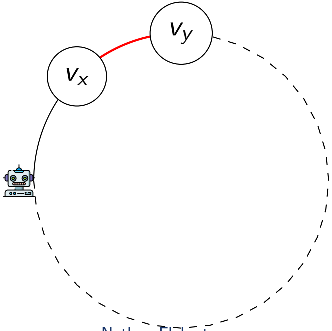

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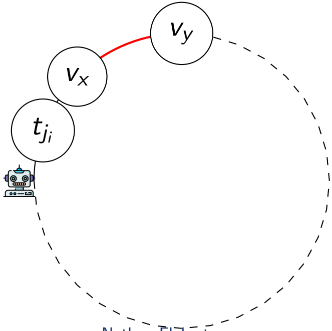

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We then adapt the partition algorithm for use on cycles by *"flattening"* the cycle.

Lemma

Given an instance of k-ROBOT SCHEDULING on a cycle $G = (V, E)$ with a set of equal duration tasks $T = \{t_1, \ldots, t_m\}$ and robots $r_1, \ldots, r_k$, there exists a fastest collision-free task-completing schedule C such that there exists some edge $e \in E$ that is not traversed by any robot in C.

The idea of our algorithm on a cycle is to iterate through all $e \in E$ such that we minimise $S[k, m]$ when applying the partition algorithm on $P = (V, E \setminus \{e\})$, this takes time $O(kmn^2)$.

Contents

Problem Formulation - Variables

- $x_{r,v,t} \in \{0, 1\}$ is 1 if robot r is at vertex v at timestep t
- $TC_{i,t} \in \{0, 1\}$ is 1 if task i is complete by timestep t
- $AC_t \in \{0, 1\}$ is 1 on the smallest timestep t where all tasks are complete.

Problem Formulation

$$\text{Maximise} \sum_{t\in[\tau]} AC_t$$

Subject To:

$$\sum_{v\in V} x_{r,v,t} = 1 \qquad \forall r \in [k], t \in [\tau] \tag{1}$$

$$x_{r,v,t} \leq \sum_{N(v)\cup\{v\}} x_{r,v,t-1} \qquad \forall r \in [k], t \in [2,\tau] \tag{2}$$

$$\sum_{r\in[k]} x_{r,v,t} \leq 1 \qquad \forall v \in V, t \in [\tau] \tag{3}$$

$$x_{r,v,t} + x_{r,v',t-1} + x_{r',v,t-1} + x_{r',v',t} \leq 3 \qquad \forall r \in [k], r' \in [k] \setminus \{r\}, (v,v') \in E, t \in [\tau] \tag{4}$$

$$TC_{i,t} \leq TC_{i,t-1} + \max_{r\in[k]} \left(\sum_{j\in[t-d_i,t]} x_{r,v_i,j}/d_i \right) \qquad \forall i \in [T], t \in [\tau] \tag{5}$$

$$TC_{i,t} \geq TC_{i,t-1} \qquad \forall i \in [T], t \in [\tau] \tag{6}$$

$$AC_t \leq \sum_{i\in[|T|]} TC_{i,t}/|T| \qquad \forall t \in [\tau] \tag{7}$$

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$$x_{r,v,t} + x_{r,v',t-1} + x_{r',v,t-1} + x_{r',v',t} \leq 3 \qquad \forall (v, v') \in E, \forall r, \forall t.$$

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- Ensures no collisions on edges

Integer Programming Constraints

$$TC_{i,t} \leq TC_{i,t-1} + \max_{r\in[k]}\left(\sum_{j\in[t-d_i,t]} x_{r,v_i,j}/d_i\right) \quad \forall i \in [T], t \in [\tau]$$

- Handles task completion.

$$TC_{i,t} \geq TC_{i,t-1} \quad \forall i \in [T], t \in [\tau]$$

- Ensures a task which is complete remains complete.

$$AC_t \leq \sum_{i\in[|T|]} TC_{i,t}/|T| \quad \forall t \in [\tau]$$

- Ensures AC is 1 iff all tasks have been completed

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- It produced an optimal solution in 11,061,661 cases (86.6%)

Partition Algorithm vs Gurobi Optimizer

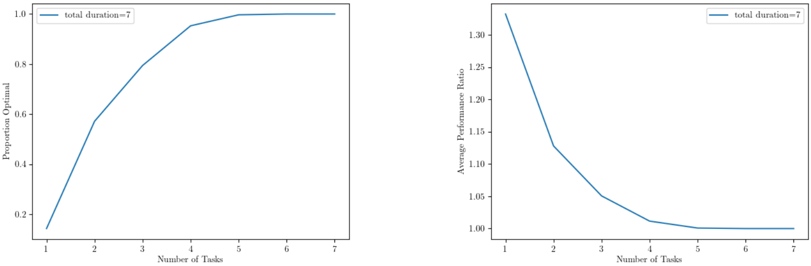

Figure 1: Comparison of the performance ratio of the Partition Algorithm to the theoretical optimal results given by our integer programming model. In this case, the tasks had a sum of durations of 7, while the number of tasks varies across the *x*-axis.

Experimental Results

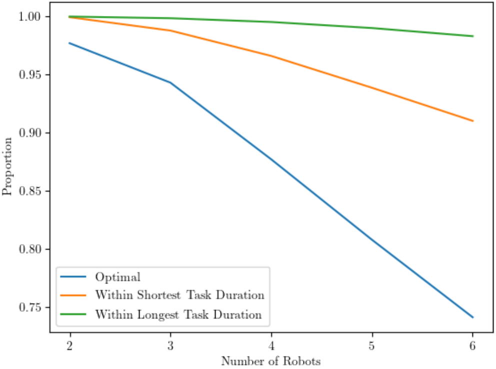

Figure 2: Comparison of proportion of output from the partition algorithm with optimal timespan, and timespan within the size of a single task from optimal for fixed $n = 7$. The plot shows the proportion decreasing as the number of robots increase

Runtime

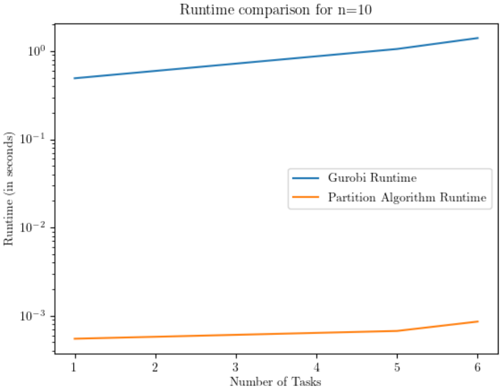

Figure 3: logplot showing average runtime of Gurobi solving the linear program and the partition algorithm solving the same instances.

Gurobi vs Partition Algorithm for Grid Graphs

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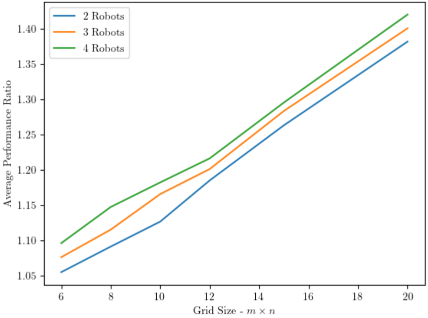

Figure 4: A plot showing the average performance ratio over all generated instances increasing as the grid size increases for 2,3 and 4 robots. In general Performance ratio is lower for fewer robots.

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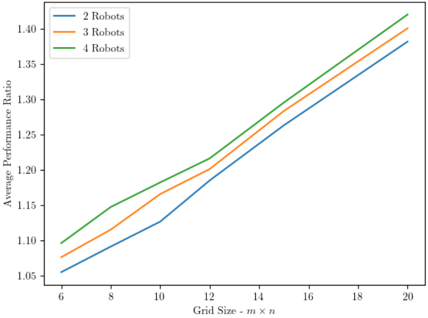

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Overall - partitioning produced an optimal result 66% of the time.

Contents

Thank You!