STRUCTURAL PARAMETERS FOR DENSE TEMPORAL GRAPHS

BCTCS

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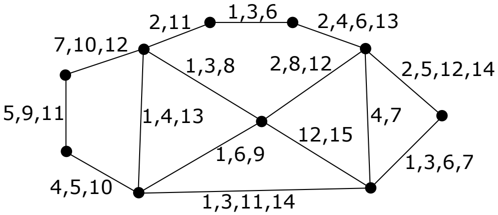
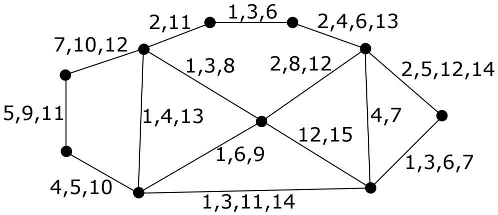
TEMPORAL GRAPHS



A temporal graph is a pair (G, λ) where G is a graph and λ maps edges of G to non-empty subsets of $\mathbb{N}$.



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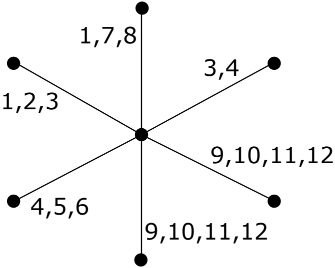
Given $e \in E(G)$ and $t \in \lambda(e)$, we call (e, t) a time-edge or edge appearance.

STAREXP
Input: A temporal graph (S_n, λ), where S_n is a star with n leaves.
Question: Does there exist a temporal walk, starting and ending at the centre of the star, that visits every vertex?



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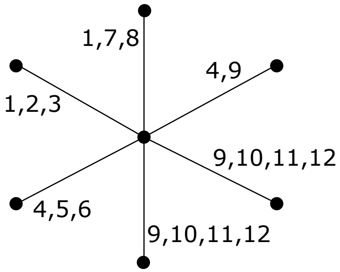

SO WE NEED PARAMETERISED APPROACHES!

How might we parameterise?

- Restrict the underlying graph
- Restrict the temporal structure
- Restrict something else: e.g. solution size

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- TEMPORAL GRAPH BURNING is NP-hard even when G is a clique or a path (Hand, 2024).

WE NEED TO CONSIDER TEMPORAL STRUCTURE

Obvious temporal parameters:

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- (vertex/edge)-interval-membership-width

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- Temporal cliquewidth

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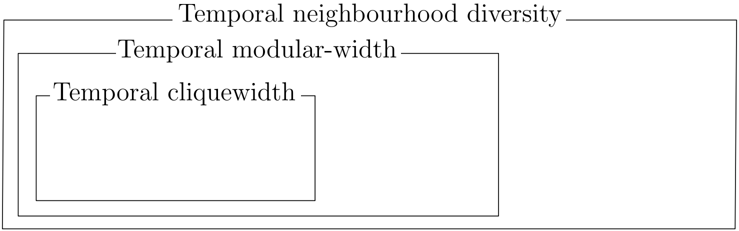
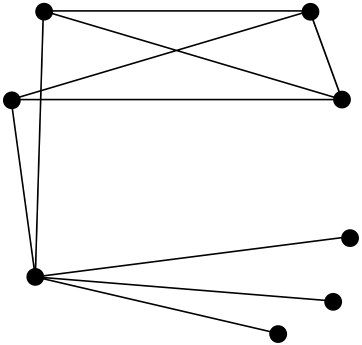


The neighbourhood diversity of a graph $G = (V, E)$ is the smallest integer k such that V can be partitioned into sets $V_1, \ldots, V_k$ with the property that, if $x, y \in V_i$ for any i then $N(x) \setminus \{y\} = N(y) \setminus \{x\}$.



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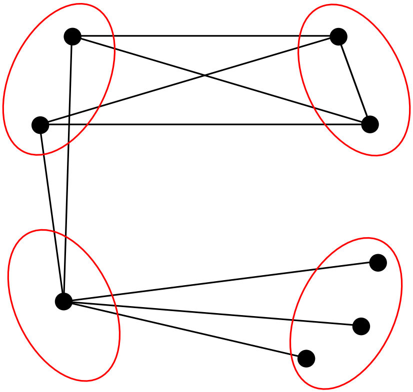
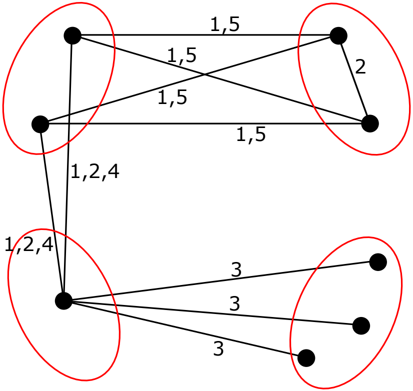


The temporal neighbourhood diversity of a temporal graph $(G = (V, E), \lambda)$ is the smallest integer k such that V can be partitioned into sets $V_1, \ldots, V_k$ with the property that, if $x, y \in V_i$ for any i then, for all times t and all vertices $z \notin \{x, y\}$, $t \in \lambda(xz)$ if and only if $t \in \lambda(yz)$.



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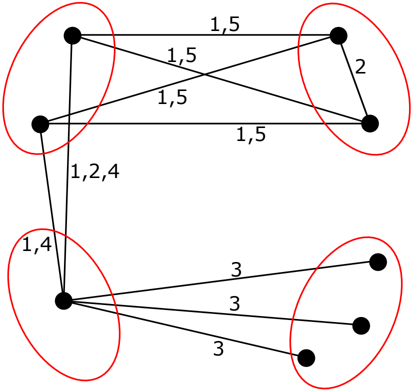
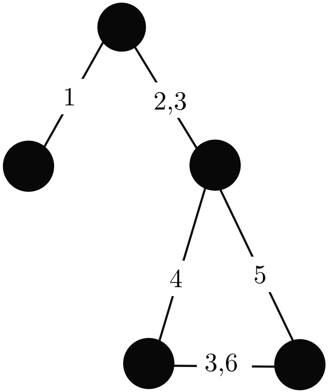


Temporal Graph Burning:



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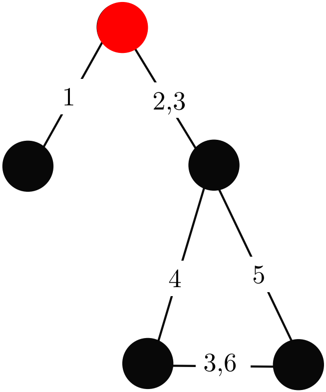
1. At time $t = 0$ a fire is placed at a chosen vertex. All other vertices are unburnt.



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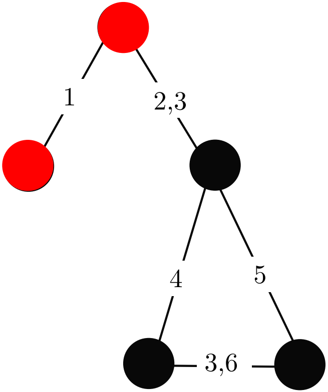

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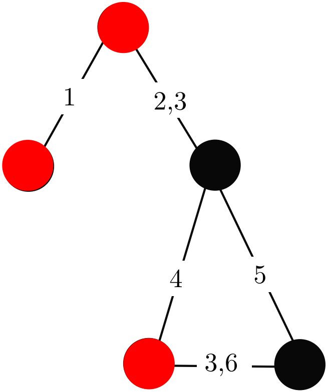

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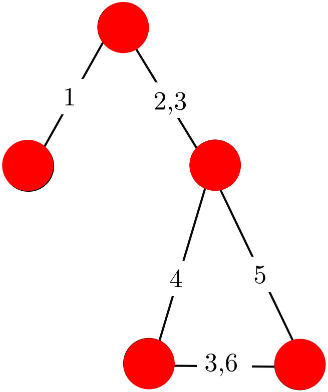
3. This process ends once all vertices are burning.



TEMPORAL GRAPH BURNING
Input: A temporal graph (G, λ) and an integer ℓ.
Question: Does there exist a successful burning strategy for (G, λ) of length less than or equal to ℓ?

Theorem
TEMPORAL GRAPH BURNING admits an FPT algorithm parameterised by the temporal neighbourhood diversity of the input graph.

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TRACTABILITY WITH TEMPORAL NEIGHBOURHOOD DIVERSITY

1. Each set of vertices in the temporal neighbourhood decomposition is either an independent set or a clique at any time.
2. Vertices in a set are either burned by being chosen to be set on fire, burned by a vertex inside the same set, or burned by a vertex in a different set.
3. If a set is an independent set and not burned by a vertex in a different set, then each vertex must be burned individually.

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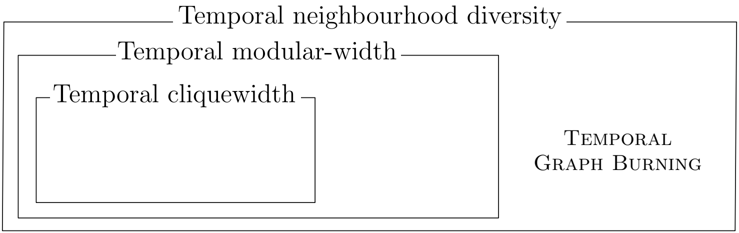
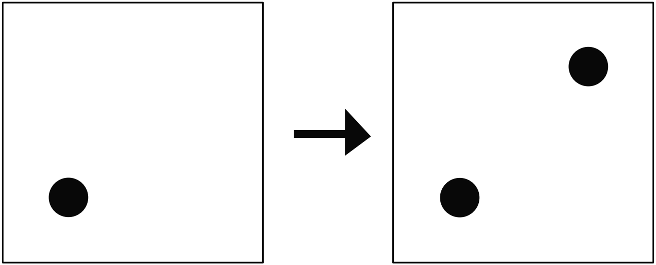
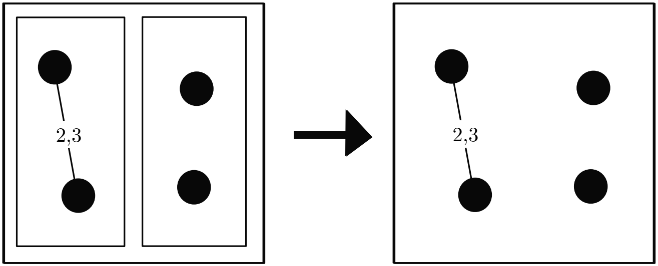


Suppose a temporal graph (G, λ) can be constructed by the algebraic expression A which uses the following operations:

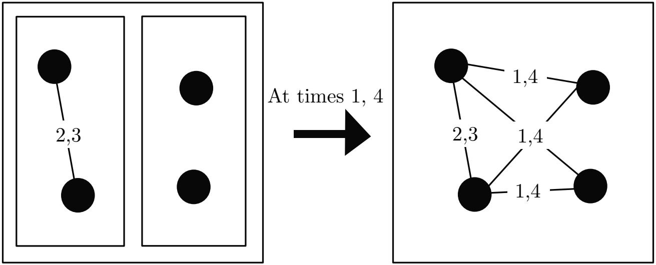
1. Creating an isolated vertex.



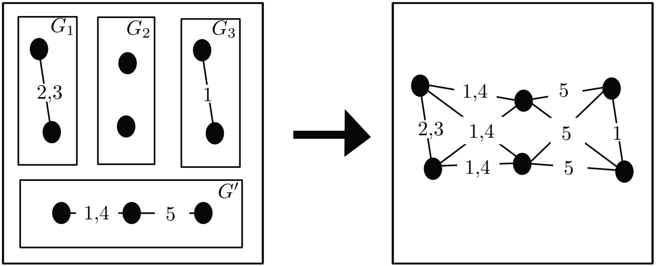
2. Taking the disjoint union of two temporal graphs.



3. Taking the complete join of two temporal graphs at a set of times T.



4. The substitution of temporal graphs $\mathcal{G}_1, \ldots, \mathcal{G}_k$ into a temporal graph $\mathcal{G}'$ with vertices $v_1, \ldots, v_k$.



The width of an expression A is the maximum number of vertices of a graph into which we perform a substitution in A.

The **temporal modular-width** of (G, λ) is this minimum width of an expression A which constructs (G, λ).

Theorem

TEMPORAL GRAPH BURNING is NP-hard even when restricted to graphs with constant temporal modular-width.

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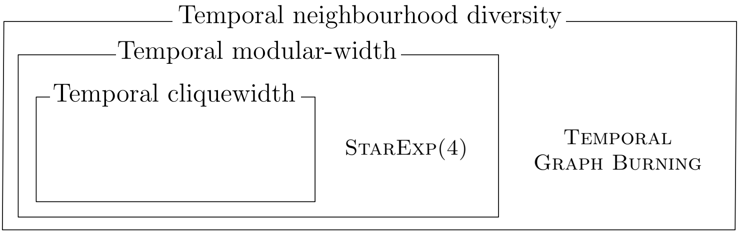
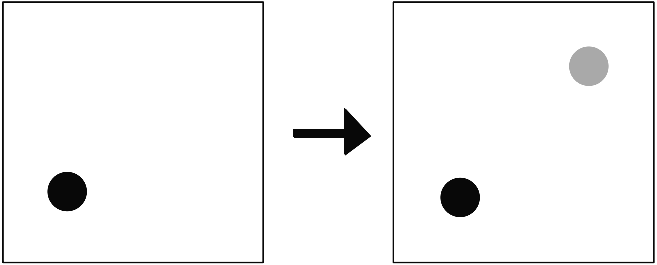
STAREXP is solvable in time $(k\tau)!(k\tau)^{O(1)}$ when the temporal modular-width of the graph is at most k and every edge appears at most τ times.

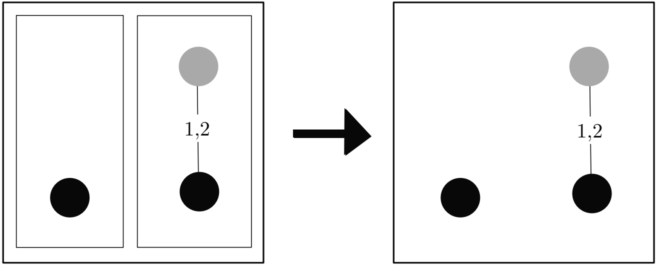

Temporal neighbourhood diversity
Temporal modular-width
Temporal cliquewidth
StarExp(4)
Temporal Graph Burning

The temporal cliquewidth of a temporal graph $\mathcal{G} = (\mathsf{G}, \lambda)$ is the number of labels required to construct $\mathcal{G}$ using only the following operations:

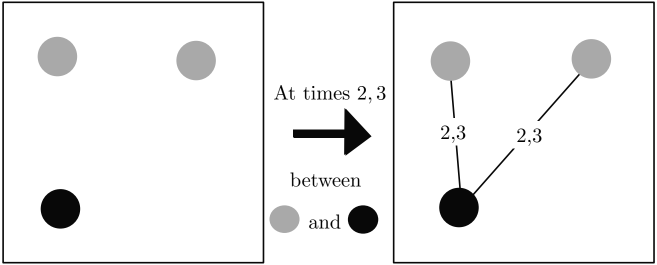
1. Creating a new vertex with label.



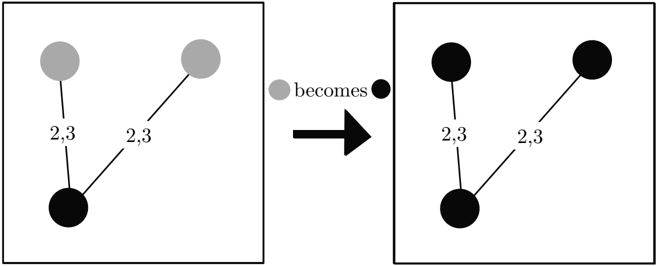
2. Taking the disjoint union of two labeled temporal graphs.



3. Adding edges to join all vertices labeled i to all vertices labeled j, where i $\neq$ j, such that all the added edges are active at the same set of times.



4. Renaming label i to label j.



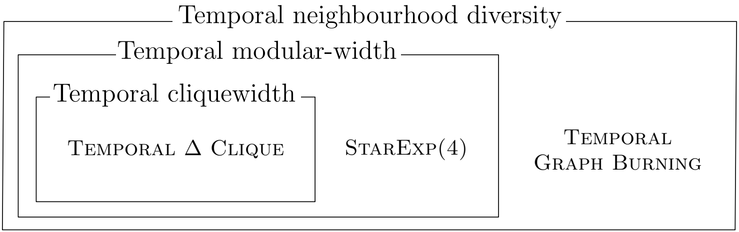
Graphs of bounded modular width cannot contain long induced paths, whereas an n-vertex path has cliquewidth 3 for arbitrarily large n.

Theorem

StarExp is NP-hard on temporal graphs with temporal cliquewidth 3.

TEMPORAL Δ CLIQUE
Input: A temporal graph $\mathcal{G} = (V, E, \lambda)$ and two integers Δ and h.
Question: Is there a set $V' \subseteq V$ of at least h vertices such that, for every $u, v \in V'$ and every window of Δ consecutive timesteps, the edge uv appears at least once in the window?

Theorem
TEMPORAL Δ CLIQUE is in FPT parameterised by the temporal cliquewidth of the input graph (provided that we are given a temporal cliquewidth construction of the input graph).


Temporal neighbourhood diversity
Temporal modular-width
Temporal cliquewidth
Temporal Δ Clique
StarExp(4)
Temporal Graph Burning

THANKS AND FUTURE DIRECTIONS

- Find more problems that are tractable parameterised by these parameters
- Is there a Courcelle-style metatheorem for temporal cliquewidth?
- Investigate the values of these parameters on real-world temporal networks

Figure 1: arxiv.org/abs/2404.19453