A Recipe for the Semantics of Reversible Programming

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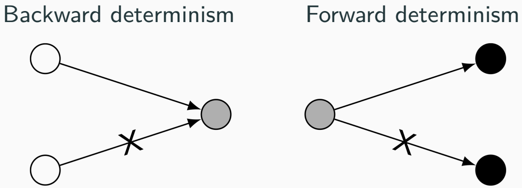
Reversible Programming

Originally

- Landauer and Bennett, 1961: Reversible Computation and Energy Dissipation.
- Reversible programs: for a program $\texttt{t}$, there is $\texttt{t}^{-1}$ such that $\texttt{t};\texttt{t}^{-1} = \texttt{skip}$.
- Applications to quantum computing.

What we do

- Reversibility, but not totality.
- Syntax for reversible functions.
- With enough expressivity.
- Through the categorical semantics.



[Kaarsgaard&Rennela21]

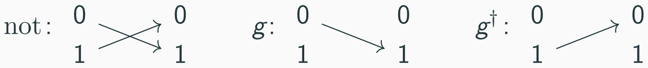
A general framework: dagger categories

Origine: functional analysis where $\langle fx \mid y \rangle = \langle x \mid f^\dagger y \rangle$.

Category $\mathbf{C}$ equipped with a functor $(-)^\dagger \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$, such that:

- On objects, $A^\dagger = A$.
- On morphisms:
  - $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$,
  - $f^{\dagger\dagger} = f$.

Example with partial injective functions between sets, here $\{0, 1\}$.



A very important class of morphisms: *partial †-isomorphism*. $ff^\dagger f = f$.

Examples of relevant dagger categories

Sets and bijections.

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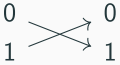

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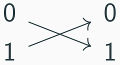

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Hilbert spaces and unitary maps.

$$|0\rangle \longrightarrow |+\rangle$$
$$|1\rangle \longrightarrow |-\rangle$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad |-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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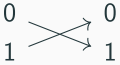
Sets and bijections. 

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Hilbert spaces and contractions.
$$h\colon \begin{array}{ccc} |0\rangle & \longrightarrow & |+\rangle \\ |1\rangle & & |-\rangle \end{array} \qquad h\,|1\rangle = 0.$$

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Hopefully, there is another way.

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Together with pattern-matching

With a sum type $\oplus$:

$$\frac{\Delta \vdash t \colon A}{\Delta \vdash \mathtt{inj}_l\, t \colon A \oplus B} \qquad \frac{\Delta \vdash t \colon B}{\Delta \vdash \mathtt{inj}_r\, t \colon A \oplus B}$$

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Our functions are then:

$$\left\{ \begin{array}{ccc} t_1 & \mapsto & t'_1 \\ t_2 & \mapsto & t'_2 \\ & \vdots & \\ t_m & \mapsto & t'_m \end{array} \right\} : A \leftrightarrow B$$

whenever $\Delta_i \vdash t_i\colon A$ and $t_j \perp t_k$, $\Delta_i \vdash t'_i\colon B$ and $t'_j \perp t'_k$.

Example

- $x\colon A \vdash t\colon C$
- $y\colon A \vdash t'\colon C$
- $t \perp t'$

$$\left\{ \begin{array}{lcl} \texttt{inj}_l\, x & \mapsto & t \\ \texttt{inj}_r\, y & \mapsto & t' \end{array} \right\} : A \oplus B \leftrightarrow C$$

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  - Compatible morphisms on their domain and codomain admit a join.
  - Sometimes, provides a nice structure on morphisms.

Examples:

- Sets and partial injective functions **PInj**.
- Hilbert spaces and contractions **Contr** (sometimes written $\mathbf{Hilb}_{\leq 1}$).

The case of inverse categories (such as PInj)

To Infinity and Beyond

Some reading: [Axelsen&Kaarsgaard16] + [Fiore04] + some calculations.

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$\longrightarrow$ A suitable inverse category **C** is parameterised **DCPO**-algebraically ω-compact.

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$$\mathtt{map}(\omega) = \textbf{fix } \mathsf{f}. \left\{ \begin{array}{ll} [\,] & \mapsto [\,] \\ h :: t & \mapsto (\omega\ h) :: (\mathsf{f}\ \ t) \end{array} \right\} : [A] \leftrightarrow [B]$$

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$$\boxed{\mathrm{fix}(F) = \sup_n\{F^n(\bot)\}}$$

Summary of the language (mandatory slide)

(Ground types)	A, B	$::=$	$\mathtt{I} \mid A \oplus B \mid A \otimes B \mid$
(Function types)	T_1, T_2	$::=$	$A \leftrightarrow B \mid$
(Unit term)	t, t_1, t_2	$::=$	$*$
(Pairing)			$\mid t_1 \otimes t_2$
(Injections)			$\mid \mathtt{inj}_l\ t \mid \mathtt{inj}_r\ t$
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(Fixed points)			$\mid f \mid \textbf{fix}\ f.\omega$
(Higher abstractions)			$\mid \lambda f.\omega \mid \omega_2\omega_1$

λ-calculus with fixed points thanks to **DCPO**-enrichment.

Expressivity

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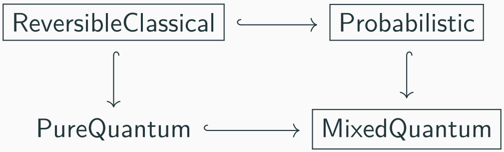
Rough summary

- Reversible Turing Machines [Axelsen&Glück11].
  - Simulate your favourite Turing machines.
- Encode RTMs in our language:
  - Alphabet & states mapped to $\mathtt{I} \oplus \cdots \oplus \mathtt{I}$.
  - Tape as lists.
  - Functions simulating one-step transition of δ.
  - Iterate until final state.

The (pure) quantum case

The quantum troubles



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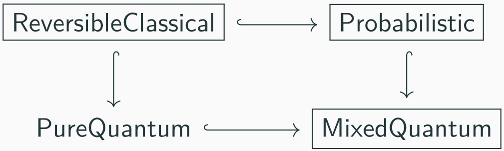

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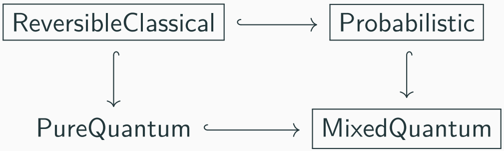

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Composition does not preserve any reasonable poset structure.

A kind of solution with techniques adapted from guarded recursion.

Computation in the topos of trees

Objects in the topos of trees are cochains in **Set**:

$$X(0) \xleftarrow[r_0]{} X(1) \xleftarrow[r_1]{} X(2) \longleftarrow \cdots$$

There is a functor $L\colon \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$, such that LX is:

$$1 \xleftarrow[!]{} X(0) \xleftarrow[r_0]{} X(1) \xleftarrow[r_1]{} X(2) \longleftarrow \cdots$$

and a natural transformation $\nu\colon \mathrm{id} \Rightarrow L$, such that ν_X is:

$$\begin{array}{ccccccccc} X(0) & \xleftarrow[r_0]{} & X(1) & \xleftarrow[r_1]{} & X(2) & \xleftarrow[r_2]{} & X(3) & \longleftarrow & \cdots \\ \downarrow ! & & \downarrow r_0 & & \downarrow r_1 & & \downarrow r_2 & & \\ 1 & \xleftarrow[!]{} & X(0) & \xleftarrow[r_0]{} & X(1) & \xleftarrow[r_1]{} & X(2) & \longleftarrow & \cdots \end{array}$$

and a family of morphisms $\mathrm{fix}_X\colon [LX \to X] \to X$.

A guarded category

Start with a dagger category **C**.

Consider the category whose objects are cochains in **C**:

$$X(0) \longleftarrow X(1) \longleftarrow X(2) \longleftarrow \cdots$$

And morphisms are natural transformations in **C**:

$$\begin{array}{ccccccc} X(0) & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y(0) & \longleftarrow & Y(1) & \longleftarrow & Y(2) & \longleftarrow & \cdots \end{array}$$

This category is enriched in the topos of trees $\mathbf{Set}^{\mathbb{N}^{\mathrm{op}}}$ (with its fixed point operator).

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It still allows for the map function.

$$\texttt{map}(\omega) = \mathbf{fix}\ \mathsf{f}.\ \left\{ \begin{array}{lcl} [\,] & \mapsto & [\,] \\ h :: t & \mapsto & (\omega\ h) :: (\mathsf{f}\ t) \end{array} \right\} : [A] \leftrightarrow [B]$$

Conclusion

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Thank you!