Decorated Para for linear quadratic regulators

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Motivation

Dynamical systems in control theory are defined over a fixed state space. ‘Flows’ are monoids generated by transition functions.

- Can we generalize monoids to categories?
- Algebraic approach to dependent action spaces, state spaces changing over time, etc.

Case study:

Linear quadratic regulators

Linear quadratic regulators: The monoid

Problem: Find a *control law* $k : X \to U$ in $\mathbf{FVec}_{\mathbb{R}}$ that given a immediate cost function $\ell = \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}$ minimises for all $x_0 \in X$ the *accumulated cost*

$$J(x_0) = \sum_i \ell(x_i, u_i) \qquad \text{s.t.} \begin{cases} x_{i+1} = f(x_i, u_i) = Ax_i + Bu_i \\ u_i = k(x_i) \end{cases}$$

Solution: Discrete algebraic **Riccati equation** ("linear algebra version of the Bellman equation"):

$$\begin{aligned} P_i &= \min_{u_i} \left\{ \ell(x_i, u_i) + (Ax_i + Bu_i)^\top P_{i+1}(Ax_i + Bu_i) \right\} \\ &= Q + A^\top P_{i+1} A - (S^\top + A^\top P_{i+1} B)(R + B^\top P_{i+1} B)^{-1}(S + B^\top P_{i+1} A) \end{aligned}$$

- P_i: Cost matrix in step i (present)
- P_{i+1}: Cost matrix in step $i+1$ (future)
- Optimal control law: $k = -(R + B^\top P_{i+1} B)^{-1}(S + B^\top P_{i+1} A)$

Linear quadratic regulators: Categorically

- Linear dynamics: $f : V \to W$ in $\mathbf{FVec}_{\mathbb{R}}$.
- Quadratic costs: Quadratic forms (symmetric matrices) in the set

$$
\begin{aligned}
\mathrm{Sym}(V) &= \left\{ \begin{array}{l} q : V \to \mathbb{R} \\ \text{s.t. } \forall \alpha \in \mathbb{R}. \forall u, v \in V. q(\alpha v) = \alpha^2 q(v) \\ \qquad q(u+v) - q(u) - q(v) \text{ is bilinear} \end{array} \right\} \\
&= \left\{ \begin{array}{l} Q : V^* \otimes V \\ \text{s.t. } \forall \alpha \in \mathbb{R}. \forall v \in V.\ (\alpha v)^\top Q(\alpha v) = \alpha^2 v^\top Q v \end{array} \right\}
\end{aligned}
$$

Costs are always positive (positive semi-definite matrices)

$$
\mathrm{Sym}_+(V) = \{ q \in \mathrm{Sym}(V) \mid \forall v \in V. q(v) \geq 0 \} = \left\{ Q \in \mathrm{Sym}(V) \mid \forall v \in V. v^\top Q v \geq 0 \right\}
$$

Structure on $\mathrm{Sym}_+(V)$

$\mathrm{Sym}(V) : \mathbf{FVec}_{\mathbb{R}}$ — pointwise addition $(p+q)(v) = p(v) + q(v)$
scalar multiplication $k \cdot q(v) = q(kv)$

$\mathrm{Sym}_+(V) : \mathbf{SMod}_{\mathbb{R}+}$ — positive cone of $\mathrm{Sym}(V)$

$\mathrm{Sym}_+(V) : \mathbf{SMod}^{\leq}_{\mathbb{R}+}$ — Löwner order (partial order, no least upper bounds)
$A \leq B$ if $B - A$ is pos. semi-definite

$$\forall v \in V. v^\top (B - A) v \geq 0$$

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$$\begin{tikzcd}
\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \arrow[dr, dash] & \\
& \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \arrow[ur, dash] \arrow[dr, dash] & & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \arrow[dl, dash] \\
& & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &
\end{tikzcd}$$

Structure on $\mathrm{Sym}_+(V)$

- $\mathrm{Sym}_+ : \mathbf{FVec}_{\mathbb{R}}^{\mathrm{op}} \to \mathbf{SMod}_{\mathbb{R}+}^{\leq}$
  - On morphisms $f : V \to W$, $\mathrm{Sym}_+(f) := f^\top(-)f : \mathrm{Sym}_+(W) \to \mathrm{Sym}(V)$ is linear (semimodule hom) and preserves order
  - 'Linear' or 'quantified' version of $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Pos}$

[1] David I. Spivak. "Generalized Lens Categories via functors $C^{op} \to Cat$". In: (Feb. 2020). arXiv: 1908.02202 [math.CT].

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- Fibrational: $\mathrm{Sym}_+ : \mathbf{FVec}_{\mathbb{R}}^{\mathrm{op}} \to \mathbf{SMod}_{\mathbb{R}+}^{\leq} \hookrightarrow \mathbf{Cat}$
  - Grothendieck construction $\int \mathrm{Sym}_+$ (aka. F-lenses[1]):
    - ★ objects $(V : \mathbf{FVec}, P : \mathrm{Sym}_+(V))$
    - ★ morphisms $(V, P) \to (W, Q)$ are $f : V \to W$ s.t. $P \leq f^\top Q f$

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Constraints via ~~Lagrangians~~ pullbacks

Lagrangian approach

Given a cost matrix $P_X \in \mathrm{Sym}_+(X)$, the state(s) $x \in X$ that minimise the cost $x^\top P_X x$ are given by the null space $\operatorname{argmin}_x x^\top P x = \ker(P)$.
Given dynamics $f : X \to Y$ (matrix A) and cost functions $P_X \in \mathrm{Sym}_+(X)$, $P_Y \in \mathrm{Sym}_+(Y)$, the *delayed rewards* problem is:

$$\begin{aligned} \operatorname{argmin}_{x \in X} \quad & J(x) = x^\top P_X x + y^\top P_Y y \\ \text{s.t.} \quad & y = Ax \end{aligned}$$

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Define $\mathcal{L}(x, \lambda, y) = J(x, y) + \lambda^\top (Ax - y)$. Then $\operatorname{argmin}_{(x,\lambda,y)}\{J(x) \mid y = f(x)\} = \ker(\nabla \mathcal{L})$.

$$\begin{aligned} \partial_x \mathcal{L}(x, \lambda, y) &= 0 \\ \partial_\lambda \mathcal{L}(x, \lambda, y) &= 0 \\ \partial_y \mathcal{L}(x, \lambda, y) &= 0 \end{aligned} \qquad \begin{bmatrix} P_X & A^\top & \\ A & 0 & -I \\ & -I & P_Y \end{bmatrix} \begin{bmatrix} x \\ \lambda \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Constraints via ~~Lagrangians~~ pullbacks

Pullback approach

Let (X, P_X) and (Y, P_Y) be objects of $\int \mathrm{Sym}_+$. The cartesian lift of $f : X \to Y$ determines $(X, f^\top P_Y f)$.

$$\pi_X \circ \ker(\nabla \mathcal{L}) = \ker(P_X + f^\top P_Y f)$$

This circumvents differentiation of $\mathcal{L}$ and the introduction of Lagrange multipliers in the Gaussian elimination algorithm, giving the optimal x^* directly.

Parametrised categories

For monoidal $(\mathcal{C}, \otimes, I)$, **Para**$(\mathcal{C})$ has objects of $\mathcal{C}$ and morphisms $X \to Y$ are $f : X \otimes U \to Y$.
For cartesian $(\mathcal{C}, \times, 1)$, **Para**$(\mathcal{C})$ is related[2] to **Span**$(\mathcal{C})$ where a morphism $X \to Y$ is $X \xleftarrow{\pi_X} X \times U \xrightarrow{f} Y$.

Example

Linear controlled dynamical systems are $X \xleftarrow{\pi_X} X \oplus U \xrightarrow{f} Y$ in $\mathbf{Span}(\mathbf{FVec}_{\mathbb{R}}) \cong \mathbf{LinSpan}_{\mathbb{R}}$.

[2] David Jaz Myers Matteo Cappuci. "Constructing triple categories of cybernetics processes". 2023.

Decorated Para

As a variation on decorated cospans[3]:

Definition

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category with a comonoid object $(1, \Delta)$, $(\mathcal{D}, +, 0)$ a symmetric monoidal category with pullbacks, and $(F, \phi) : (\mathcal{D}, +, 0)^{\mathsf{op}} \to (\mathcal{C}, \otimes, I)$ a lax monoidal contravariant functor. The category F**Span** of F-decorated spans consists of

- Objects of $\mathcal{D}$
- Morphisms $X \to Y$ are equiv. classes of spans $X \xleftarrow{l} N \xrightarrow{r} Y$ and 'structure elements' $s : 1 \to FN$.
- Composition: Spans by pullback. Structure elements by:

$$1 \xrightarrow{\Delta} 1 \otimes 1 \xrightarrow{s \otimes t} FN \otimes FM \xrightarrow{\psi_{N,M}} F(N + M) \xrightarrow{F[\pi_N, \pi_M]} F(N +_Y M)$$

[3] Brendan Fong. "The Algebra of Open and Interconnected Systems". PhD thesis. University of Oxford, 2016. DOI: 10.48550/ARXIV.1609.05382.

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Example

For $\mathrm{Sym}_+ : (\mathbf{FVec}_{\mathbb{R}}, \oplus, \mathbb{R}^0) \to (\mathbf{SMod}^{\leq}_{\mathbb{R}+}, \oplus, \emptyset)$, the category Sym_+ **Span** with comonoid object $(\mathbb{R}, \Delta)$ consists of

- Objects: State spaces $X : \mathbf{FVec}_{\mathbb{R}}$
- Morphisms: $X \xleftarrow{\pi_X} X \oplus U \xrightarrow{f} Y$ and a cost matrix $P_{XU} : \mathrm{Sym}_+(X \oplus U)$, denoted $(\pi_X, f, P_{XU}) : X \to Y$.
- Identity: $X = X = X$ and $0_X : \mathrm{Sym}_+(X)$.

The functorial Riccati equation

Recall the discrete algebraic Riccati equation (DARE)

$$P_i = Q + A^\top P_{i+1} A - (S^\top + A^\top P_{i+1} B)(R + B^\top P_{i+1} B)^{-1}(S + B^\top P_{i+1} A)$$

Theorem

The discrete algebraic Riccati equation defines a functor $\mathrm{DARE} : (\mathsf{Sym}_+ \, \mathbf{Span})^{\mathsf{op}} \to \mathrm{Mfd}_\infty$.

- It's smooth
- Preserves identity:

$$\mathrm{DARE}(\mathsf{id}_X, \mathsf{id}_X, 0_X)(P_X) = 0_X + \mathsf{id}_X^\top P_X \, \mathsf{id}_X = P_X$$

- Preserves compositionality (by Gaussian elimination...):

$$\mathrm{DARE}(\pi_X, f, P_{XU}) \circ \mathrm{DARE}(\pi_Y, g, P_{YV})(P_Z) = \mathrm{DARE}(g \circ (f \oplus 1_V), P_{\mathsf{gr}(f)V})(P_Z)$$

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Thank you!