On hypergraph colouring variants and inapproximability

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BCTCS 2025

This talk covers joint work with Tamio-Vesa Nakajima, Marcin Wrochna, and Stanislav Živný

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- Warm-up: Graphs
  - Graph colouring
  - Approximate graph colouring
  - Algorithms and hardness results
- Hypergraphs
  - Hypergraph colouring variants
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  - Hardness results
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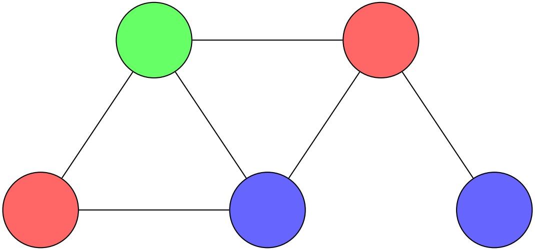
Graph colourings

Definition

A *k-colouring* of a graph (V, E) is an assignment of colours $c(v) \in [k]$[1] to the vertices $v \in V$ such that for every edge $\{u, v\} \in E$ we have

$$c(u) \neq c(v)$$



[1] For any integer k, we write $[k]$ for the set $\{0, 1, \ldots, k-1\}$.

Complexity of Graph Colouring

Fact

Finding a 2-colouring of a 2-colourable graph can be done in polynomial time (e.g. using breadth-first search)

Theorem (Karp75)

Finding a k-colouring of a k-colourable graph is NP-hard for every $k \geq 3$.

Approximate Graph Colouring

Example: 3vs5-colouring

Given a 3-colourable graph, find a 5-colouring

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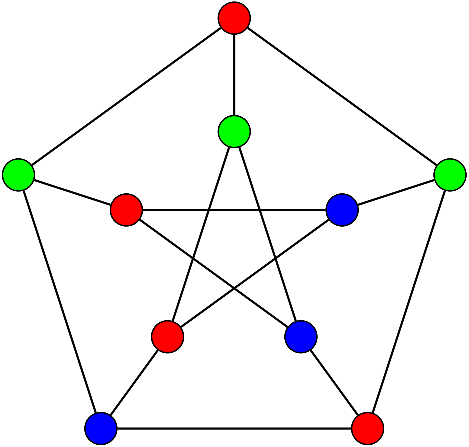

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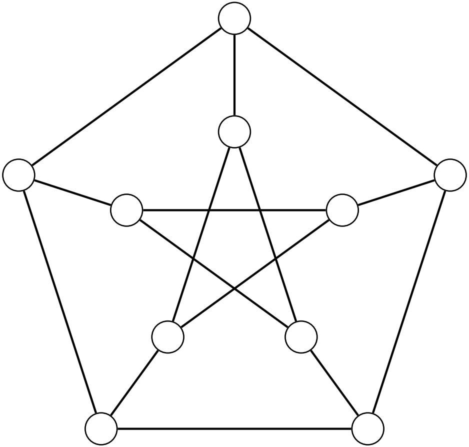

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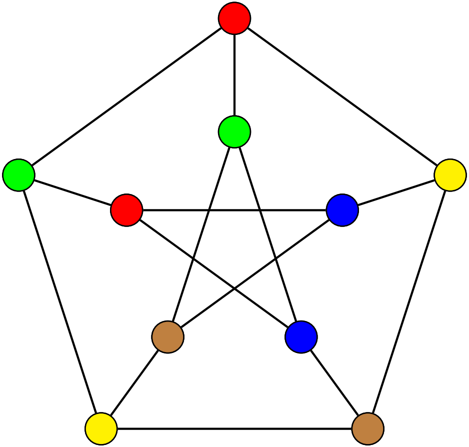

Approximate Graph Colouring hardness

Notation

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However, we have no (unconditional) proof of the hardness of PCSP(K_3, K_6).

Approximate Graph Colouring algorithms

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Note that there is a large gap between 5 and $O(n^{0.19747})$.

Hypergraphs

Definition

A *hypergraph* is a pair $(V, \mathcal{E})$, where V is the vertex set and $\mathcal{E}$ is the hyperedge set. Each hyperedge $e \in \mathcal{E}$ is a multiset of vertices (meaning vertices can appear multiple times in a single hyperedge). A hypergraph is *r-uniform* if all the hyperedges have size r.

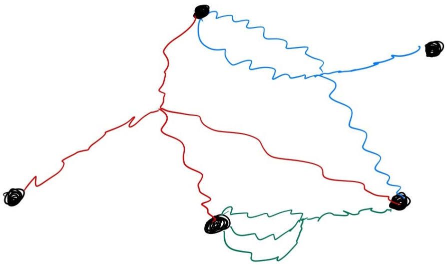

Figure: Example of a 4-uniform hypergraph with 5 vertices and 3 hyperedges

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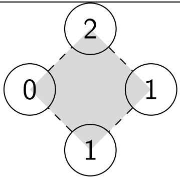
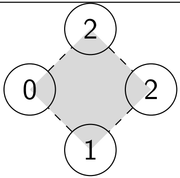
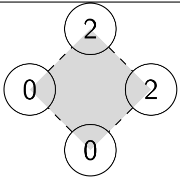
Arity 2 (graphs): these are all the same.
Arity 3: nonmonochromatic and conflict-free are the same.
Arity 4 and above: these are different notions.

Hypergraph colouring variants

Hyperedge	nonmonochromatic	conflict-free	linearly ordered
	Yes	Yes	Yes
	Yes	Yes	No
	Yes	No	No

Approximate hypergraph colourings

- We can define approximate versions for the problem of finding a nonmonochromatic/conflict-free/linearly ordered hypergraph colouring.

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- We can frame these as promise constraint satisfaction problems: e.g. conflict-free $k$vsℓ-colouring for r-uniform hypergraphs is written as $\mathrm{PCSP}(\mathbf{CF}_k^r, \mathbf{CF}_\ell^r)$

Hardness results

In a recent preprint (N**V**WŽ25 to appear in ICALP), we proved that:

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- The second result was shown for $r \geq \ell - k + 4$ by (NŽ22).
- The polynomial-time solvable case is due to being able to encode this problem as a system of linear equations mod 2.
- The rest of this talk will be dedicated to a (sketch) proof of the last result for $r = 4$ and $k = 3$.

Polymorphisms

Definition

A *polymorphism* of arity n of (CF_3^4, CF_ℓ^4) is a function $f : [3]^n \to [\ell]$ such that

$$\text{whenever every column of } \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \\ d_1 & \cdots & d_n \end{pmatrix} \text{ has a unique entry,}$$

$$\begin{pmatrix} f(a_1, \cdots, a_n) \\ f(b_1, \cdots, b_n) \\ f(c_1, \cdots, c_n) \\ f(d_1, \cdots, d_n) \end{pmatrix} \text{ has a unique entry.}$$

Example

Any projection is a polymorphism.

Avoiding sets

Definition

Let $f : A^n \to B$ and $T \subseteq B$. A *T-avoiding set* for f is a set $X \subseteq [n]$ such that for for any input $\vec{v} \in \{0,1\}^n$ with $\vec{v}|_X \equiv 1$, we have $f(\vec{v}) \notin T$.

$$f(\underbrace{1, 1, \dots, 1, 1}_{X}, 0 \text{ or } 1, 0 \text{ or } 1, \dots, 0 \text{ or } 1) \notin T$$

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For $t \in \mathbb{N}$, we call a set X t-avoiding for f if it is T-avoiding for f for some subset $T \subseteq B$ of size t.

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Using the algebraic approach to PCSPs, we can find a sufficient condition for NP-hardness in terms of avoiding sets, by reducing from label cover (a known NP-hard problem).

Hardness condition

Theorem (NV**WŽ25)**

Let $(\mathbf{A}, \mathbf{B})$ be a PCSP template with $\{0,1\} \subseteq A$ and $\ell = |B|$. Suppose that there exist constants $N, \{\alpha_t\}_{t=1}^{\ell}, \{\beta_t\}_{t=1}^{\ell}$ such that every $f \in \mathrm{Pol}(\mathbf{A}, \mathbf{B})$ has the following properties:

1. f has a 1-avoiding set of size $\leq \beta_1$.
2. If f is of arity $\geq N$ and has a disjoint family of $> \alpha_t$ many t-avoiding sets, all of size $\leq \beta_t$, then f has a $(t+1)$-avoiding set of size $\leq \beta_{t+1}$.

Then, $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.

Hardness condition

Theorem (N$\mathbf{V}$WŽ25) (less technical version)

PCSP($\mathbf{A}, \mathbf{B}$) is NP-hard if every polymorphism f of $(\mathbf{A}, \mathbf{B})$ has the following properties (the bounds can not depend on the arity of the polymorphism):

1. f has a 1-avoiding set of bounded size.
2. If f has many disjoint t-avoiding sets of bounded size,then these can be used to show that f has a $(t+1)$-avoiding set of bounded size.

Hardness condition

Theorem (N$\mathbf{V}$WŽ25) (Applied to conflict-free 3vsℓ-colouring)

Conflict-free 3vsℓ-colouring is NP-hard if every polymorphism f of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ has the following properties (the bounds can not depend on the arity of the polymorphism):

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We will show that every polymorphism of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ has a 1-avoiding set. The second property can be shown by a similar argument, hence proving hardness of $\mathrm{PCSP}(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$.

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Note

A 1-avoiding set for f an n-ary polymorphism of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ is a subset X of $[n]$, such that there exists some $b \in [\ell]$ such that for every $\vec{v} \in \{0,1\}^n$ with $\vec{v}|_X \equiv 1$, $f(\vec{v}) \neq b$

Kneser Graphs

Definition

The *Kneser graph* is defined as $\mathrm{KG}(n, h) = ([n]^{(h)}, E)$, where $S \sim_E T$ if and only if $S \cap T = \emptyset$ ($[n]^{(h)}$ means the family of subsets of $[n]$ of size h).

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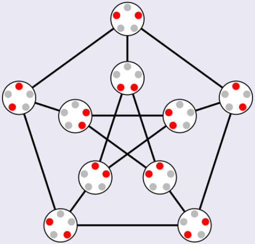
Example

$\mathrm{KG}(5, 2)$ is the Petersen graph



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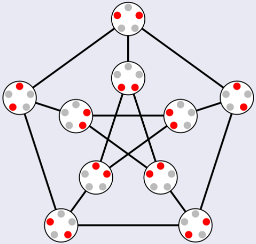

Theorem (Lovász78)

The chromatic number of $\mathrm{KG}(n, h))$ is $n - 2h + 2$.

Polymorphisms of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ have small 1-avoiding sets

Theorem

Any polymorphism f of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ has a 1-avoiding set of size $\leq \ell$.

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Proof

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Theorem

Any polymorphism f of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ has a 1-avoiding set of size $\leq \ell$.

Proof

Let n be the arity of f. Note that we may assume $n > \ell$.
Set $h = \lceil \frac{n-\ell}{2} \rceil$. Consider the Kneser graph $\mathrm{KG}(n, h)$, and colour each vertex S by $f(\vec{2}_S)$. $\chi(\mathrm{KG}(n,h)) = n - 2h + 2 > \ell$, so there exist disjoint sets $S, T \in [n]^{(h)}$ such that $f(\vec{2}_S) = f(\vec{2}_T)$. Let $X \coloneqq [n] \setminus (S \cup T)$.
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Note

A 1-avoiding set for f an n-ary polymorphism of $(\mathbf{CF}_3^4, \mathbf{CF}_\ell^4)$ is a subset X of $[n]$, such that there exists some $b \in [\ell]$ such that for every $\vec{v} \in \{0,1\}^n$ with $\vec{v}|_X \equiv 1$, $f(\vec{v}) \neq b$.

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$$\begin{pmatrix} \overbrace{\begin{matrix} 2 & \cdots & 2 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0/1 & \cdots & 0/1 \end{matrix}}^{\text{S}} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ 2 & \cdots & 2 \\ 0 & \cdots & 0 \\ 0/1 & \cdots & 0/1 \end{matrix}}^{\text{T}} & \overbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{matrix}}^{\text{X}} \end{pmatrix}$$

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Further work: Approximate LO-colouring

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- Can we prove hardness of e.g. $\mathrm{PCSP}(\mathbf{LO}_2^3, \mathbf{LO}_3^3)$?