A fibration of traces over configurations

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Context

Goal in denotational semantics: reconcile

- game semantics (for functional languages) with
- presheaf models (for concurrent languages).

For this to remotely make sense, need to link strategies with presheaves.

Link between strategies and presheaves (thanks Sam)

- Traces $\approx$ Plays.
- From strategies to presheaves:
  - prefix-closed sets of traces;
  - functors $\mathrm{Traces}^{op} \longrightarrow 2$, where 2 is the poset $0 \leqslant 1$;
  - functors $\mathrm{Traces}^{op} \longrightarrow$ sets.
- Intuitively, presheaves may accept a trace in several ways.
- Example on the board if not all comfortable with this.

Innocent presheaves

- Presheaves are thus generalised strategies.
- Crucial in game semantics: innocence.
- Extends to presheaves!
  - Start from presheaves on traces with prefix ordering.
  - Refine traces to a proper category of multi-agent traces.
  - Incorporate game-semantical innocence:

    each agent reacts only according to its neighbourhood.

    Here maybe I can say it: it's a sheaf semantics...
- For CCS and π: innocent presheaves $\rightsquigarrow$ full abstraction w.r.t. fair testing.

This talk

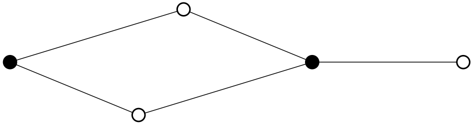
General goal

Generalise our constructions, by reworking them properly.

This talk: focus on a crucial aspect related to fibrations.

1. Give a feel for the category of CCS multi-agent traces and innocent presheaves.
2. Construct it from a double category, using the fact that the `vertical codomain' functor is a fibration.
3. Show how the latter follows in part from a factorisation system.
4. Point out the aspects that we hope to improve.

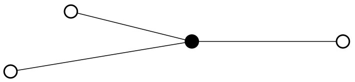
Configurations



- ● ≈ agent.
- ○ ≈ channel.
- Edges : `agent knows channel'.

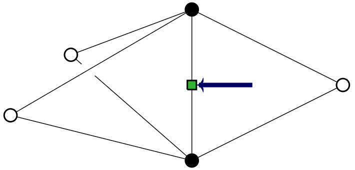
Example (atomic) action: input

Initial and final configurations are the same, e.g.



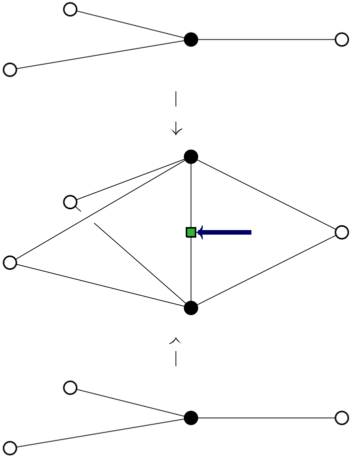
- But: actions are not a mere binary relation (initial, final configuration).
- Instead: cospans initial $\rightarrow$ stuff $\leftarrow$ final.
- What stuff? A kind of higher-dimensional graph.

Higher-dimensional graph for the input action



- The arrow indicates on which channel the input occurs.
- One such graph for each arity (here 3).
- Cute formal definition as a cat $\mathscr{C}$ of finite presheaves (cf. graphs).

Cospan for the input action

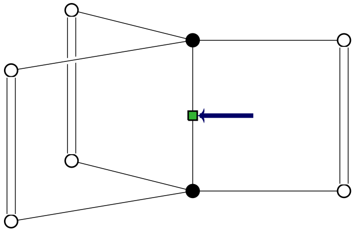


final configuration

stuff

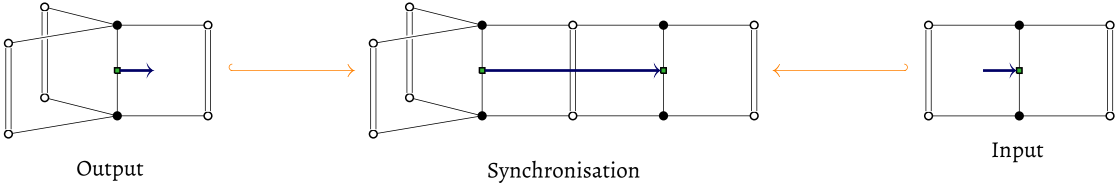
initial configuration

drawn for conciseness as:



Actions: input/output

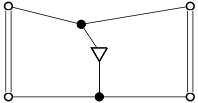
Using the previous convention:



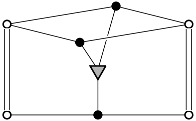
Orange arrows: morphisms of cospans

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Y' \arrow[d] \\
T \arrow[r] & T' \\
X \arrow[r] \arrow[u] & X' \arrow[u]
\end{tikzcd}$$

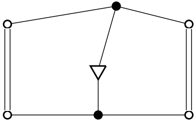
Actions, continued



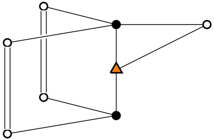
Left fork



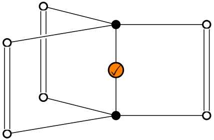
Fork



Right fork



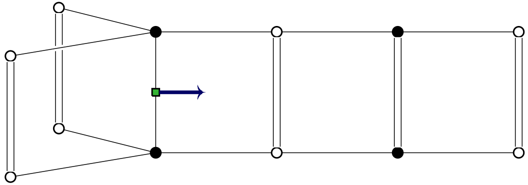
Channel creation



Tick

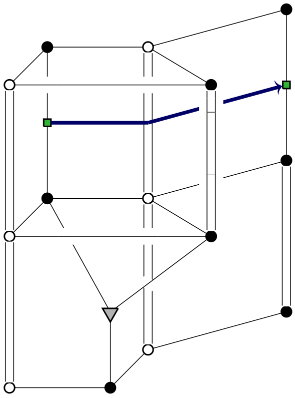
Local vs. global actions

- Until now, actions were local: only involved agents were shown.
- Global actions obtained by embedding into larger configurations.
- E.g.:



Multi-agent traces

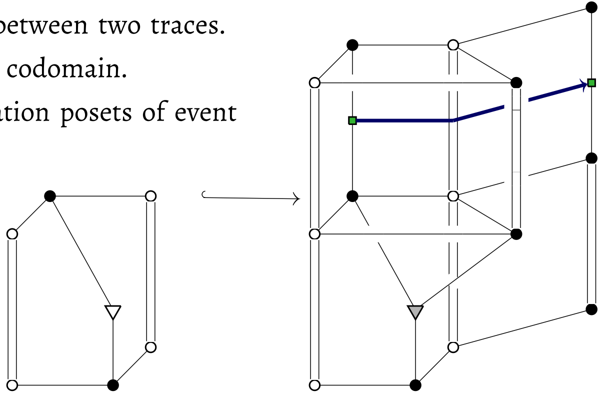
Obtained by piling up global actions:



Concurrent: no particular ordering between remote actions.

Category of traces: $\mathcal{T}$

- Spatial and temporal embedding:
  - new moves after the final configuration,
  - new agents (there from the beginning).
- Possibly several morphisms between two traces.
- Ex: if two agents fork in the codomain.
- Otherwise, close to configuration posets of event structures.

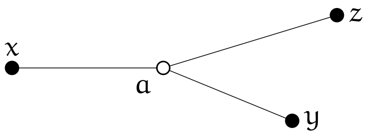


Relativise to configuration X: $\mathcal{T}_X$.

(Non-innocent) presheaves over configuration X

Let us consider presheaves $\mathcal{T}_X^{op} \to$ sets.

Too general: consider the configuration



and the presheaf

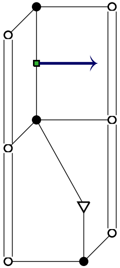
- accepting $x \to y$,
- accepting outside $\to z$,
- but refusing $x \to z$.

Agents x and z should not be allowed to choose with whom they synchronise.

Innocent presheaves

Views: let $\mathcal{V}_X \subseteq \mathcal{T}_X$ consist of histories of exactly one agent.

Example:



Definition 1. Innocent presheaves

Presheaf $\mathcal{V}_X^{op} \longrightarrow$ sets.

Problem: no obvious inclusion innocent $\subseteq$ non-innocent (not the same base cat).

Innocent $\rightarrowtail$ non-innocent

By right Kan extension:

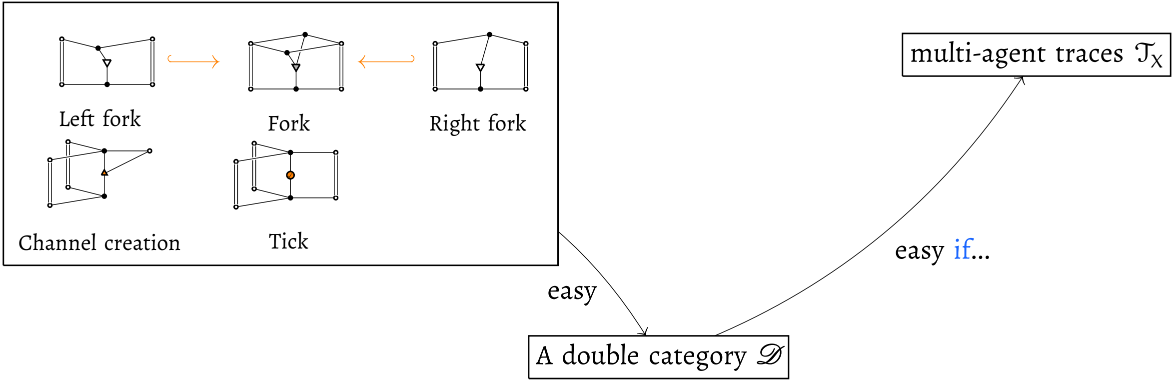
$$\begin{tikzcd} \mathcal{V}_X^{op} \arrow[rr, hook, "i^{op}"] \arrow[dr, "S"'] & \arrow[d, phantom, "\cong"] & \mathcal{T}_X^{op} \arrow[dl, "\bar{S} = \mathrm{ran}_{i^{op}}(S)"] \\ & \mathrm{sets} & \end{tikzcd}$$

Explicit formula

- General : $\bar{S}(T) = \int_{V \in \mathcal{V}_X} S(V)^{\mathcal{T}_X(V,T)}$.
- Boolean case; T accepted iff all its views are : $\bar{S}(T) = \bigwedge_{\{(V \xrightarrow{\alpha} T) \in \mathcal{T}_X\}} S(V)$.
- General case:

  way of accepting T = compatible family of ways of accepting its views.

Focus now



... the vertical codomain functor $\mathscr{D}_H \to \mathscr{D}_h$ is a fibration!

And that holds thanks to a factorisation system.

Reminder on (pseudo) double cats

1. Objects, vertical / horizontal arrows, (double) cells
2. Vertical ($\bullet$) / horizontal ($\circ$) compositions of arrows and cells.
3. The interchange law: both ways of parsing

$$
\begin{tikzcd}
Z \arrow[r, "l"] \arrow[d, "T"'] \arrow[dr, phantom, "\beta"] & Z' \arrow[r, "l'"] \arrow[d, "T'"'] \arrow[dr, phantom, "\beta'"] & Z'' \arrow[d, "T''"'] \\
Y \arrow[r, "k"] \arrow[d, "S"'] \arrow[dr, phantom, "\alpha"] & Y' \arrow[r, "k'"] \arrow[d, "S'"'] \arrow[dr, phantom, "\alpha'"] & Y'' \arrow[d, "S''"'] \\
X \arrow[r, "h"] & X' \arrow[r, "h'"] & X''
\end{tikzcd}
$$

commute, i.e., $(\alpha' \bullet \beta') \circ (\alpha \bullet \beta) = (\alpha' \circ \alpha) \bullet (\beta' \circ \beta)$.

The double category $\mathscr{D}^0$

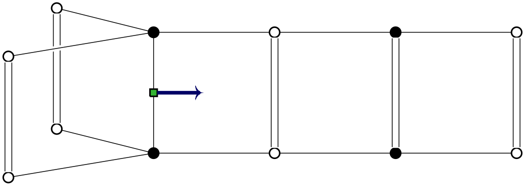
- Objects: configurations.
- Horizontal arrows: morphisms $X \to Y$ in $\mathscr{C}$.
- Vertical arrows: cospans in $\mathscr{C}$.
- Cells:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ T \arrow[r] & T' \\ X \arrow[r] \arrow[u] & X' \arrow[u] \end{tikzcd}$$

- Vertical composition of cells by universal property of pushout.

The double category $\mathscr{D}$

Restrict vertical arrows to the subbicategory of Cospan($\widehat{\mathscr{C}}$) generated by the (global versions of the) given cospans, e.g.,



$\mathscr{D} \rightsquigarrow \mathcal{T}_X$

$\mathcal{T}_X$ is almost $\mathscr{D}_H$, the category of vertical arrows of $\mathscr{D}$.

It has:

- Objects: vertical morphisms of $\mathscr{D}$.
- Arrows $T \to T'$ (up to some boring twist): diagrams

$$
\begin{tikzcd}
Z \arrow[r, "k"] \arrow[d, "S"'] & Y' \arrow[dd, "T'"] \\
Y \arrow[d, "T"'] \arrow[r, phantom, "\alpha"] & \\
X \arrow[r, "h"] & X'
\end{tikzcd}
$$

Intuition

- Extend T with some actions (S).
- Embed the result into T'.

Composition in $\mathcal{T}_X$

Plan:

$$
\begin{tikzcd}
C \arrow[r, "l"] \arrow[d, "W"'] \arrow[dr, phantom, "\gamma"] & Z' \arrow[r, "k'"] \arrow[d, "S'"] \arrow[dddr, phantom, "\beta"] & Y'' \arrow[ddd, "T''"'] \\
Z \arrow[r, "k"] \arrow[d, "S"'] \arrow[ddr, phantom, "\alpha"] & Y' \arrow[dd, "T'"'] & \\
Y \arrow[d, "T"'] & & \\
X \arrow[r, "h"] & X' \arrow[r, "h'"] & X''
\end{tikzcd}
$$

- Orange part should be a Cartesian lifting of S' along k.
- In our double cat $\mathscr{D}$: Cartesian lifting follows from factorisation.
- W should be a trace again, i.e.:

Traces should be stable under restriction in $\mathscr{D}^0$.

Factorisation systems (Mac Lane, 1950; Freyd & Kelly, 1972)

Consider a category $\mathcal{C}$.

Definition 2.

Let $f \perp g$ iff for all commuting squares

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

there exists a unique lifting h making both triangles commute.

Extends to sets of morphisms:

- $\mathcal{E} \perp \mathcal{M}$ iff for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$, $e \perp m$;
- $\mathcal{E}^{\perp} = \{m \mid \forall e \in \mathcal{E}.\ e \perp m\}$.
- ${}^{\perp}\mathcal{M} = \{e \mid \forall m \in \mathcal{M}.\ e \perp m\}$.

Systèmes de factorisation (Freyd et Kelly, 1972)

Definition 3. Factorisation system on $\mathcal{C}$

$(\mathcal{L}, \mathcal{R})$ s.t.

- $\mathcal{R} = \mathcal{L}^{\perp}$,
- $\mathcal{L} = {}^{\perp}\mathcal{R}$,
- any f factors as $r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

Proposition 1.

$\mathcal{L}$ stable under pushout and composition.

$\mathcal{R}$ stable under pullback and composition.

Cofibrant generation (Bousfield, 1977)

Theorem 1.

Consider a set $\mathcal{C}$ of morphisms in a locally presentable category. Then

$$\left({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp}\right)$$

forms a factorisation system.

- Only the factorisation axiom is hard (small object argument).
- Very well explained on Joyal's Catlab.

http://ncatlab.org/joyalscatlab/show/Factorisation+systems

Application

Let's consider a simpler setting, where the only action looks like

- Possible interpretation of this action: agent jumps (a.k.a. the Pixar game)...
- So we have only one `generating' cospan, in Gph:

$$[0] \xrightarrow{s} [1] \xleftarrow{t} [0],$$

  where [1] is the graph with one edge $s \to t$.
- s is the final configuration.

Application

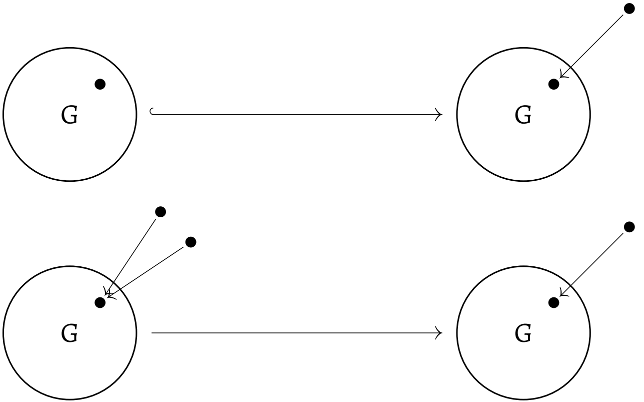
We take for $\mathcal{C}$ the singleton $\{[0] \xrightarrow{t} [1]\}$, i.e.,

the injection of the initial configuration into the action.

- Bousfield's construction $\rightsquigarrow$ factorisation system $(\mathscr{V}, \mathscr{O})$ (vertical, other).
- In this case, $\mathscr{V}$ consists of
  - transfinite composites of
  - pushouts of
  - guys in $\mathcal{C}$ and their codiagonals.

Application

Essentially, $\mathscr{V}$ consists of composites of morphisms of one of the shapes



Lifting cospans

We consider the sub-double category $\mathscr{D}^1 \subseteq \mathscr{D}^0$ consisting of cospans
$$\begin{tikzcd} X' \arrow[d] \\ T \\ X \arrow[u, "\mathscr{V} \ni v"] \end{tikzcd} .$$
We have $\mathscr{D} \subseteq \mathscr{D}^1 \subseteq \mathscr{D}^0$.

Lifting in $\mathscr{D}^1$

Lifting (v, f) along k :

- factor $v \circ o$ as $o' \circ v'$;

$$\begin{tikzcd}
Y' \arrow[r, "o'' \in \mathscr{O}"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f"] \\
V \arrow[r, "o' \in \mathscr{O}"] & U \\
Y \arrow[u, "\mathscr{V} \ni v'"] \arrow[r, "k"'] & X \arrow[u, "v \in \mathscr{V}"']
\end{tikzcd}$$

- take the pullback of f and o'.

Universal property of lifting

$$
\begin{tikzcd}
Z' \arrow[dr] \arrow[drr, bend left] \arrow[dd] & & \\
& Y' \arrow[r] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & X' \arrow[dd] \\
W \arrow[dr] \arrow[drr, bend left] & & \\
& V \arrow[r, "o' \in \mathscr{O}"'] & U \\
Z \arrow[uu, "\mathscr{V} \ni v''"] \arrow[dr] \arrow[drr, bend left] & & \\
& Y \arrow[r] \arrow[uu] & X \arrow[uu]
\end{tikzcd}
$$

Result

Theorem 2.
Traces are fibred over configurations.

Proof sketch.

- Proved abstractly:

  Cospans $X \xrightarrow{v} T \xleftarrow{f} X'$ are fibred over configurations.
  Or, the codomain functor $\mathscr{D}^1_H \to \mathscr{D}^1_h$ is a fibration.

- For traces: more requirements, proved by hand, case by case for CCS and π.

Would-be proof sketch

We could hope for more: traces $\subseteq$ cospans $X \xrightarrow{v} T \xleftarrow{w} X'$, where

- X and X' are configurations,
- $v \in \mathscr{V}$,
- $w \in \mathscr{W} = {}^{\perp}(\mathscr{S}^{\perp})$, for $\mathscr{S} = \{[0] \xrightarrow{s} [1]\}$.

Better-looking, false in general

Cospans (v, w) fibred over configurations.

- Traces even more specific than such cospans.
- What makes it work in our cases?

Double factorisation systems (Tholen & Pultr)

Promising direction?

Definition 4.

Two factorisation systems $(\mathscr{V}, \mathscr{O})$ and $(\mathscr{W}, \mathscr{H})$ such that $\mathscr{V} \subseteq \mathscr{W}$.

Let $\overline{X} = {}^{\perp}(X^{\perp})$.

- Here: $\mathcal{C} \subseteq \mathcal{C} \cup \mathscr{S}$ hence $\mathscr{V} = \overline{\mathcal{C}} \subseteq \overline{\mathcal{C} \cup \mathscr{S}} = \mathscr{W}$.
- Main result: any arrow factors as $A \xrightarrow{\mathscr{V}} B \xrightarrow{\mathscr{W} \setminus \mathscr{V}} C \xrightarrow{\mathscr{H}} D$.

Double factorisation systems

Any arrow factors as $A \xrightarrow{\mathscr{V}} B \xrightarrow{\mathscr{W} \setminus \mathscr{V}} C \xrightarrow{\mathscr{H}} D$.

- Good points:
  - Morphisms of configurations are in $\mathscr{H}$.
  - $\mathscr{H} ; \mathscr{V} \subseteq \mathscr{V} ; \mathscr{H}$.
  - We get useful information on the bottom square.
- Bad points:
  - Unclear what makes the top square work.
  - $\overline{\mathscr{S}}$ not taken seriously.

Conclusion

- Multi-agent traces $\leadsto$ innocent presheaf semantics for CCS and π-calculus.
- Construction via a `fibred' double category $\mathscr{D}$.
- Cartesian lifting by factorisation.
- More to understand, maybe by making double factorisation systems more symmetric.

Longer-term perspectives

- Scale the approach to calculi with passivation, functional languages...
- Tighten our axiomatisation.
- Link with exotic settings like cellular automata.
- Double categories as abstract rewriting systems with space?