Fibrational Units of Measure

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- Class of representations is a *dimension*, e.g., Length (L), Mass (M), Time (T)
- Normally pick out base units for dimensions, e.g., SI Base units include kg, m, s, K,...etc
  $Derived\ Units\ kgm^{-2}s^{-2}$

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- $\times : \forall u_1.\forall u_2.num(u_1) \to num(u_2) \to num(u_1 \cdot u_2)$

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- Introduces type system and relational semantics
- ... But no category theory.

Outline

- Type System for Units of Measure
- Categorical Semantics
- Examples and Theorems
- Parametricity

A Type System for Units of Measure

Unit Context $\Delta = u_1, ..., u_n$

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$$\frac{\Delta, u \vdash T\ \mathrm{Type}}{\Delta \vdash \forall u.T\ \mathrm{Type}}$$

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Typing Context $\Gamma = x_1 : T_1, ..., x_m : T_m$

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and

$$\frac{\Delta \vdash \Gamma \text{ ctxt} \quad \Delta, u, \Gamma \vdash t : T}{\Delta; \Gamma \vdash \Lambda u.t : \forall u.T} \qquad \frac{\Delta \vdash e \quad \Delta, \Gamma \vdash t : \forall u.T}{\Delta; \Gamma \vdash te : T[e/u]}$$

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- $\mathcal{B}$ - Unit contexts and expressions
- $\mathcal{E}$ - Types and terms

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Definition

We call (p, G, num) a *UoM-fibration*.

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  Codomain, Suboject and Relations fibration over Set are λ_1-fibrations with simple products.

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  Codomain, Suboject and Relations fibration over Set are λ_1-fibrations with simple products.
  - Choose Abelian group object, G
  - Choose object in fibre above G
- *Unit Erasure Semantics* $(p : \mathcal{E} \to 1, *, \mathrm{num})$
  e.g. $\mathcal{E} =$ cpo and num $= \mathbb{Q}_\perp$

Theorems About UoM-Fibrations

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Theorem

- $(p : \mathcal{E} \to \mathcal{B}, G, X)$ *UoM-fibration*
- $\mathcal{A}$ *finite products*
- $F : \mathcal{A} \to \mathcal{B}$ *product preserving functor*
- $G' \in \mathcal{A}$ *an Abelian group object with*

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The Ab-Set *fibration is a λ_1-fibration with simple products. Hence, for choices $\mathcal{G} \in \text{Ab}$, $\text{num} \in \text{Ab-Set}_{\mathcal{G}}$*

$$(p : \text{Ab-Set} \to \text{Ab}, G, \text{num}) \quad \textit{is a UoM-fibration}$$

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- $p : \mathcal{E} \to \mathcal{B}$ *is a fibration with* $\mathcal{E}_X := [X] \to \mathcal{D}$ *and hence reindexing is given by precomposition*
    - *i.e., for any* $f : X \to Y \in \mathcal{B}, \quad f^*(\phi : [Y] \to \mathcal{D}) = \phi \circ [f]$.

Then, the reindexing of any projection map $\pi_X : X \times Y \to X$ *has a right adjoint* $\pi_X^* \dashv Ran_{[\pi]}$ *, which satisfies the Beck-Chevalley condition.*

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- AND right adjoints are given by right Kan extensions
- Then quantification satisfies BC
- We use this to show the Ab-Set fibration is a UoM-fibration

Results in the Ab-Set Fibration

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Lemma

Suppose $u \vdash S, T$ Type, *then*

$$|[\![\forall u.S \to T]\!]| \cong Nat([\![S]\!], [\![T]\!])$$

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Proof.

By end formula for a Kan extension. $\square$

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Use previous lemma to see $[\![t]\!] \in G\text{-Set}(\mathrm{num}(u), \mathrm{num}(u^n))$

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Use previous lemma to see $[\![t]\!] \in G\text{-}\mathrm{Set}(\mathrm{num}(u), \mathrm{num}(u^n))$
Naturality gives result. $\square$

Results in the Ab-Set Fibration ctd...

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$$Rel(\mathcal{E})_n = \{(n, B, P) \mid B \in \mathcal{B},\ P \in \mathcal{E}_{R(n) \times B \times B}\}$$

Parametric UoM-Fibrations ctd...

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Theorem
$(r : \mathit{Rel}(\mathcal{E}) \to \mathsf{L}_{\mathsf{Ab}}, 1, \mathsf{num})$, *for a choice of* num, *is a UoM-fibration.*

Parametric UoM-Fibrations ctd...

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- ...write thesis...

Thanks for listening.

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