

# Fibrational Units of Measure

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- Class of representations is a *dimension*, e.g., Length ( $L$ ), Mass ( $M$ ), Time ( $T$ )
- Normally pick out base units for dimensions, e.g., SI Base units include  $kg, m, s, K, ...etc$   
*Derived Units  $kgm^{-2}s^{-2}$*

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- $\times : \forall u_1. \forall u_2. num(u_1) \rightarrow num(u_2) \rightarrow num(u_1 \cdot u_2)$

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- ... But no category theory.



# Outline

- Type System for Units of Measure
- Categorical Semantics
- Examples and Theorems
- Parametricity

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$$\frac{\Delta \vdash \Gamma \text{ ctxt} \quad \Delta, u, \Gamma \vdash t : T}{\Delta; \Gamma \vdash \lambda u. t : \forall u. T} \qquad \frac{\Delta \vdash e \quad \Delta, \Gamma \vdash t : \forall u. T}{\Delta; \Gamma \vdash te : T[e/u]}$$



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We call  $(p, G, \text{num})$  a *UoM-fibration*.

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- *Unit Erasure Semantics* ( $p : \mathcal{E} \rightarrow 1, *, \text{num}$ )  
e.g.  $\mathcal{E} = \text{cpo}$  and  $\text{num} = \mathbb{Q}_\perp$

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- $(p : \mathcal{E} \rightarrow \mathcal{B}, G, X)$  UoM-fibration
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**Then  $(F^*p, G', (G', X))$  is a UoM-fibration.**

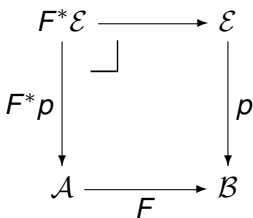
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$$F(1) = G$$

$$\begin{array}{ccc} F^*\mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow F^*p & \lrcorner & \downarrow p \\ L_{Ab} & \xrightarrow{F} & B \end{array}$$

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*The Ab-Set fibration is a  $\lambda_1$ -fibration with simple products. Hence, for choices  $\mathcal{G} \in \text{Ab}$ ,  $\text{num} \in \text{Ab-Set}_{\mathcal{G}}$*

$(p : \text{Ab-Set} \rightarrow \text{Ab}, \mathcal{G}, \text{num})$  *is a UoM-fibration*

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- $p : \mathcal{E} \rightarrow \mathcal{B}$  is a fibration with  $\mathcal{E}_X := [X] \rightarrow \mathcal{D}$  and hence reindexing is given by precomposition
  - i.e., for any  $f : X \rightarrow Y \in \mathcal{B}$ ,  $f^*(\phi : [Y] \rightarrow \mathcal{D}) = \phi \circ [f]$ .

Then, the reindexing of any projection map  $\pi_X : X \times Y \rightarrow X$  has a right adjoint  $\pi_X^* \dashv \text{Ran}_{[\pi]}$ , which satisfies the Beck-Chevalley condition.

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- AND right adjoints are given by right Kan extensions
- Then quantification satisfies BC
- We use this to show the Ab-Set fibration is a UoM-fibration

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## Proof.

By end formula for a Kan extension. □

# Results in the Ab-Set Fibration ctd...

## Lemma

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$(r : \text{Rel}(\mathcal{E}) \rightarrow \text{L}_{\text{Ab}}, 1, \text{num}),$  for a choice of  $\text{num}$ , is a UoM-fibration.

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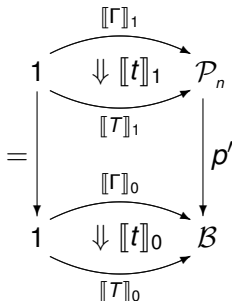
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- ...write thesis...

Thanks for listening.

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