Compositional Game Theory

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- **Question:** What is Game Theory?
  - How to make decisions in eg, finance, scheduling algorithms
  - Use Nash equilibria in non-cooperative games
- **Claim:** Game theory is too concrete
  - Uses non-structural, reductive measures, eg payoff matrices.
  - Category theory turns meta-structure into actual structure.
- **Compositionality:** Operators build big games from small games
  - Lift results about parts of a game to the whole games
  - Better mathematics and better software for games.

Part I: Simple Games

- **Defn:** A *basic game* consists of
  - A set of actions A the player can take, and a set U of utilities
  - A function $f : A \to U$ assigning to each action, a *utility*

- **Defn:** Optimal actions/equilibria for a simple game are

$$\mathsf{Eq}(A, U, f) = \mathsf{argmax}\ f = \{a \in A \mid (\forall a' \in A) fa \geq fa'\}$$

- **Question:** Is this definition correct for a game?

$$f : A_1 \times A_2 \to U_1 \times U_2$$

- **Motivation:** Two academics face a choice
  - Each is under pressure to report bad behaviour of the other to the authorities who seek evidence to discipline the academics.
  - They can cooperate with each other, or defect $\Rightarrow A = \{C, D\}$
  - Utilities are given by $f : A \times A \to Z \times Z$

$$\begin{array}{rr} f(C,C) = (0,0) & f(D,C) = (1,-3) \\ f(C,D) = (-3,1) & f(D,D) = (-2,-2) \end{array}$$

- **Conclusion:** The best strategy for each player is to defect!
  - Rather depressing for utopians! Assumptions: no communication, no future cost for bad behaviour etc.

- **Motivation:** Simple game equilibria doesn't compute the optimal strategy in the prisoner's dilemma

- **Defn:** A 2-player game is
  - Sets of actions A_1, A_2 and utilities U_1, U_2 of utilities
  - A function $f : A_1 \times A_2 \to U_1 \times U_2$ assigning to each pair of actions, a pair of utilities

- **Defn:** Optimal actions/equilibria for a 2-player game are given by $\mathsf{Nash} \subseteq A_2 \times A_2$

$$(a_1, a_2) \in \mathsf{Nash}\ f \quad \text{iff} \quad \begin{array}{l} a_1 \in \mathsf{argmax}\ (\pi_1 \circ f(-, a_2)) \\ \wedge a_2 \in \mathsf{argmax}\ (\pi_2 \circ f(a_1, -)) \end{array}$$

- **Key Idea:** Nash equilibria are given as primitive.
  - This is not a compositional definition as the definition is not derived from equilibria for simpler games
  - It is simply postulated as reasonable, justified empirically.
- **Question:** Is there no operator which combines two 1-player games into a 2-player game?
  - And defines the equilibria of the derived game via those of the component games.
- **Remark:** Of course this is difficult as optimal moves for one game may not remain optimal when that game is incorporated into a networked collection of games.

- **Defn:** A *utility-free game* consists of
  - A set A of moves, a set U of utilities and an equilibria function $\mathsf{E} : (A \to U) \to \mathsf{P}A$.
  - The set of utility-free games with actions Y and utilities U is written $\mathsf{UF}_A U$

- **Key Idea:** These games leave the utility function abstract
  - The equilibria is given for *every* potential utility function
  - And its not always argmax, eg El Farrol bar game

- **Defn:** Let $G_1 \in \mathsf{UF}_{A_1}U_1$ and $G_2 \in \mathsf{UF}_{A_2}U_2$ be UF-games.
  - Their monoidal product is the UF-game

$$G_1 \otimes G_2 : \mathsf{UF}_{A_1 \times A_2}(U_1 \times U_2)$$

    with equilibrium function

$$(y_1, y_2) \in \mathsf{E}_{G_1 \otimes G_2} k \quad \text{iff} \quad \begin{array}{l} y_1 \in \mathsf{E}_{G_1}(\pi_1 \circ k(-, y_2)) \quad \wedge \\ y_2 \in \mathsf{E}_{G_2}(\pi_2 \circ k(y_1, -)) \end{array}$$

- **Thm:** Let $G = (A_1, A_2, U_1, U_2, k)$ be a simple 2-player game. Define the utility-free games

$$G_1 = (A_1, U_1, \mathsf{argmax}) \quad G_2 = (A_2, U_2, \mathsf{argmax}).$$

  Then $(y_1, y_2) \in \mathsf{Nash}_G$ iff $(y_1, y_2) \in \mathsf{E}_{G_1 \otimes G_2} k$

- **Key Idea:** CGT is possible. Don't hardwire a specific utility.

Part II: Complex Games

- **Motivation:** Simple games possess limited structure, and hence support limited operators
  - More operators $\Rightarrow$ more compositionality
  - Lets develop a more complex model!

- **Example:** Lets place a bet
  - I have a bank balance. I might have different strategies. These factors decide on my bet which I give to the bookmaker
  - The bookmaker has a variety of strategies to deal with my bet. When the event is finished, he returns my winnings

Open Games are Typed

- **Types:** Let X, Y, S, R be sets. Think of X as the game's state.
  - Y is move or other observable action
  - R is utility which the environment produces from a move
  - S is coutility which the system feeds into the environment
- **Examples:** X is my bank balance, the bet that the bookie must react to. External factors affecting our decisions
  - Y is my bet or the action the bookie takes
  - R is my winnings or the utility gained from the move
  - S is the coutility fed back into the system, eg the bookie sends me my winnings.

- **Defn** An open game $G : (X, S) \to (Y, R)$ is defined by
  - A set Σ of strategies
  - A play function $P : \Sigma \times X \to Y$
  - A coutility function $C : \Sigma \times X \times R \to S$
  - An equilibrium function $E : X \times (Y \to R) \to \mathsf{P}\Sigma$

- **Example:** Prisoners Dilemma PD : $(1, 1) \to (M \times M, Z \times Z)$ where $M = \{C, D\}$.
  - Two rounds of prisoners dilemma?

- **Via Lenses:** A *lens* $L : (X, S) \to (Y, R)$ is a map $f : X \to Y$ and $g : X \times R \to S$

- An open game $G : (X, S) \to (Y, R)$ is a set Σ and for each $\sigma \in \Sigma$
  - A lens $G_\sigma : (X, S) \to (Y, R)$
  - A predicte $\mathsf{E}_\sigma \subseteq ((1, 1) \to (X, S)) \times ((Y, R) \to (1, 1))$

- **Via Interaction Structures and Indexed Containers**: The algebra becomes easier if we use dependent types:

$$S \;\longleftarrow^{C}\; R \;\to\; Y \;\to\; \Sigma \;\to\; X$$

- **Assume:** Given open games

$$G : (X, S) \to (Y, R) \quad \text{and} \quad G' : (X', S') \to (Y', R')$$

- **Define:** Construct an open game

$$G \otimes G' : (X \times X', S \times S') \to (Y \times Y', R \times R')$$

- **Abstraction:** Now we can define a monoidal category of open games
  - Objects are pairs of sets (X, S)
  - Morphisms $(X, S) \to (Y, R)$ are open games

- **Composition:** This requires composition. Given open games

$$G : (X, S) \to (Y, R) \quad \text{and} \quad H : (Y, R) \to (Z, T)$$

construct an open game

$$H \circ G : (X, S) \to (Z, T)$$

- **Achievements:** A new model of game theory
  - New paradigms — Compositionality
  - New concepts — Coutility
  - New Techniques — String diagrams

- **Future Work:** Much more to do
  - More operators, more categories, more algorithms
  - Translate into better software
  - Applications: smart contracts, energy grids, blockchains