Tangent Categories Mini-notes

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Abstract

These are notes taken on occasion of a mini-course (6×1 hour lectures) on tangent categories by Geoffrey Cruttwell at the University of Strathclyde in October 2026. The mini-course covers the basic definition of tangent categories, a detailed look at three motivating examples (the tangent categories of smooth affine manifolds, affine schemes and commutative rings), as well as the basic constructions involving vector bundles, vector fields, differential equations and connections.

1 Introduction

Differential geometry is understood as the study of motion and change and their associated spaces. It has myriad applications in physics, but also in machine learning. In the past decades, it was discovered that the ideas of differential geometry can be fruitfully applied to other settings, such as algebraic geometry. This suggests that the core constructions of differential geometry are "substrate independent", and so *tangent categories* is the theory of substrate-independent differential geometry, providing us a minimal/axiomatic categorical way of doing differential geometry.

The first person to define tangent categories was Jiri Rosicky in 1984 [1]. However, he was concerned about how difficult it would be to prove more complicated results and promptly abandoned it, until the notion was rediscovered by Robin Cockett and Geoffrey Cruttwell [2] in 2012. Since then, lots of additional work has gone into developing differential geometry in this axiomatic setting of tangent categories, as well as in finding examples of tangent categories.

(Thanks Geoff for the great lectures, for reading through these notes and for suggesting comprehensive feedback!)

1.1 Intended Audience and Notational Conventions

The reader is expected to be familiar with categories, functors and natural transformations, as well as pullbacks. Composition of maps is typically written in *diagrammatic order* as fg for $f : X \to Y$ followed by $g : Y \to Z$.

For the examples, the reader is expected to be familiar with multivariate calculus (at least to know what the Jacobian is) as well as the basic theory of commutative rings and algebras over a field, as well as their ideals/quotients.

1.2 Outline (and Correspondence with the Lectures)

(Section 2) With some basic intuition coming from differential geometry (it suffice that the reader knows how to draw pictures of donuts and spheres and whatnot), we will arrive at the definition of tangent categories, which roughly speaking is a category $\underline{\mathbb{X}}$ equipped with an endofunctor $T : \underline{\mathbb{X}} \to \underline{\mathbb{X}}$ sending a "manifold" $M \in \mathrm{obj}(\underline{\mathbb{X}})$ to its space of tangent vectors TM. For actual examples and computations we will work in the tangent category $\underline{\mathsf{Smooth}}$ of *affine smooth manifolds* (whose objects are open subspaces $U \subseteq \mathbb{R}^n$ and morphisms are smooth functions). This more-or-less corresponds to

the content from the first two lectures, but they have been remixed and blended a bit so that we get to play with concrete examples sooner.

(Section 3) We will solidify the intuition by providing another geometric example of tangent categories, namely the tangent category $\underline{\mathsf{cAlg}}_R^{\mathsf{op}}$ of *affine schemes* over a base commutative ring R. This corresponds to the content from the third lecture and also some of the fourth lecture.

(Section 4) As a counter-balance to the geometric intuition, we will also exhibit an algebraic example, namely the tangent category of R-algebras $\underline{\mathsf{cAlg}}_R$ (note: there is no $(-)^{\mathsf{op}}$ as compared to the above!) This corresponds to a bit of content from the fourth lecture, but there it was left as an exercise to work out the details. This is my attempt at working out the details.

(Section 5) With a good grasp on the intuition behind the definition, we will define the *differential bundles* in a tangent category, which are the tangent category analogues of vector bundles. All the TM will be (by construction) instances of differential bundles. These are one of the main concepts of differential geometry from which a lot of the theory is built out of. This is the majority of the content from the fourth lecture.

(Section 6) We work out the basic theory and construction of vector fields. In particular, the operation of the *Lie bracket* on vector fields is motivated and defined for tangent categories (this requires that the tangent bundle be an Abelian group, not merely a commutative monoid). Importantly, this was where Jiri Rosicky had mixed success and got stuck. This is content from the fifth lecture.

(Section 7) This is a short section working out the definition of differential equations in a tangent category, demonstrating the richness of tangent categories. This content is from the sixth lecture.

(Section 8) Finally, we end by discussing the theory of *connections* which comes from the end of the sixth lecture.

1.3 Further Pointers to the Literature

1.3.1 The Theory of Tangent Categories

For more details on the basic definition and basic theory of *tangent categories* (e.g. differential objects), as well as the theory of tangent (join) restriction categories and their *manifold completion*, you can consult the original paper of Cockett and Cruttwell [2]. For the theory of differential bundles and their associated fibrations, see [3]. For connections, the theory was first worked out for differential bundles [4] but there is a recent preprint which works out their theory on arbitrary maps $q : E \to M$ [5]. The theory of differential equations was worked out in [6].

There is a recent high-level introduction and survey by Marcello Lanfranchi [7] which (in my opinion) complements this set of notes really well, and additionally also expounds more on the *philosophy* of working with tangent categories as a theory of spaces. Tangent categories understand spaces as any object that allows you to situate yourself within and make local moves "locally", as represented by the tangent vectors.

1.3.2 Examples of Tangent Categories

For more details on the tangent category of schemes and how the theory of tangent categories looks here, consult [8], [9] and [10]. It may be helpful to have with you Geoff's notes on algebraic geometry [11] by your side!

For the tangent category of diffeological spaces, see [12].

2 The Definition of Tangent Category

The main example of a tangent category is the category of smooth manifolds. For this entire document, you will *not* be told what a smooth manifold M formally is, but it suffice to think of them as these surfaces which "don't have sharp edges or boundaries", or somewhat more formally, where the surface looks like an open subset of $\mathbb{R}^n$ locally around any point $x \in M$. Here are some examples, followed by some non-examples, and hopefully you can extrapolate a rough sense of what we mean.

Example 2.0.1. Examples of manifolds are:

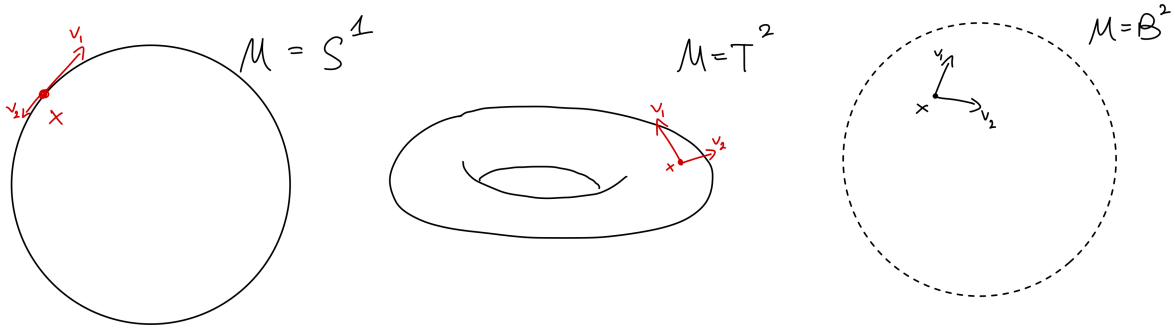
- the (un-filled) circle S^1
- the sphere S^2
- the n-dimensional spheres $S^n = \{x \in R^{n+1} \mid \|x\| = 1\}$
- the open balls $B^n = \{x \in R^{n+1} \mid \|x\| < 1\}$
- the torus/donut T^2

Non-examples of manifolds:

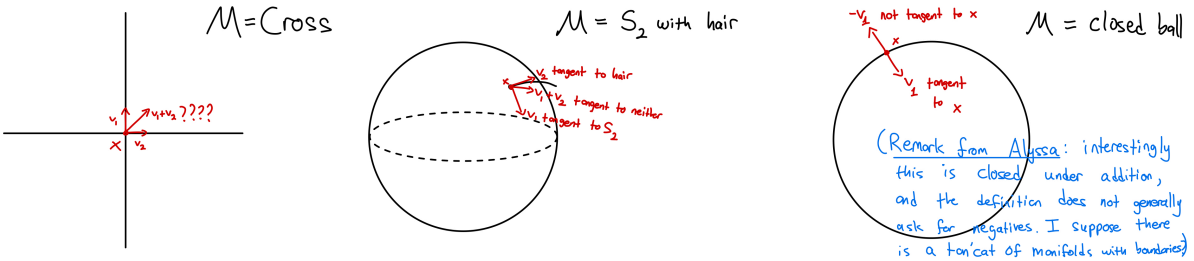
- the sphere S^2 but where we forgot to shave off one hair, because of the hair's pesky root point.
- the cross $\{(x, y) \in R^2 \mid x = 0 \text{ or } y = 0\}$, because of that annoying point $(0, 0)$ in the middle.
- the closed balls $\{x \in R^{n+1} \mid \|x\| \leq 1\}$, because of the boundary points.

Pictures are just below.

Because the surface is smooth, it makes sense to talk about vectors v_1 or v_2 of some magnitude which are *tangent* to the surface at a particular point $x \in M$ (see picture below). These form a *vector space* $T_x M$, whose dimension matches the dimension of the space[1], because if we add two tangent vectors we get another vector tangent to the surface.



Compare this with our non-examples, where addition of two tangent vectors may not be tangent again.



Remark. You might ask why we want to think about these tangent vectors. Basically, they let us do calculus, for example you can take limits and hence derivatives by "observing the limit as the tangent

[1] so e.g. $T_x S^1$ has dimension 1, $T_x S^2$ has dimension 2, and $T_x \mathbb{R}^n$ has dimension n.

vectors shrink to 0". Now, unless our tangent vectors have enough structure, its rather annoying to do calculus.

Now, our basic object of study are manifolds, so we want to arrange all the T_xM (ranging over x) into a single manifold TM. This manifold will somehow look like M except we've "inflated" or "blown-up" every point into an entire vector space. It's a bit hard to draw except for the simplest examples, but hence we get an assignment $M \mapsto TM$.

Now as category theorists we must obviously ask "is this assignment functorial?" The answer is yes, but since I'm not going to tell you what a smooth manifold really is we instead inspect a smaller example using *affine* smooth manifolds. If smooth manifolds are spaces that locally look like open subspaces of $\mathbb{R}^n$, then the affine ones are the ones that actually *are* just open subspaces of $\mathbb{R}^n$. Then we can use kindergarten multivariate calculus to work out what the functoriality looks like[2].

2.1 Baby's Smooth Manifolds (AKA Affine Manifolds)

Definition 2.1.1. The category Smooth has as objects the open subsets $U \subseteq \mathbb{R}^n$, and the maps being smooth (i.e. infinitely differentiable) functions $f : U \to V$. We will write U_n to denote an object $U \subseteq \mathbb{R}^n$ of dimension n.

The tangent space of U_n is easy to describe, unlike the general case for smooth manifolds. It is just $TU_n := U_n \times \mathbb{R}^n$, and an element $(x, v) \in TU_n$ is drawn as in the pictures we have been drawing so far as a vector v sticking out from the point x treated as origin. The action of this functor on a smooth map $f : U_n \to V_m$ is to take the *directional derivative* at a particular tangent vector v, so

$$Tf : U_n \times \mathbb{R}^n \to V_m \times \mathbb{R}^m \tag{1}$$

$$Tf : (x, v) \mapsto (f(x), (\nabla f)(x) \cdot v) \tag{2}$$

where

Definition 2.1.2. the *Jacobian* or *gradient* $(\nabla f)(x)$ of a smooth function $f : U_n \to V_m$ at $x \in U_n$ is the $m \times n$ matrix of partial derivatives for the function f, i.e.

$$\begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \cdots & \frac{\partial f_0}{\partial x_{n-1}} \\ \vdots & & \vdots \\ \frac{\partial f_{m-1}}{\partial x_0} & \cdots & \frac{\partial f_{m-1}}{\partial x_{n-1}} \end{bmatrix}. \tag{3}$$

Example 2.1.3. If $U = \mathbb{R}^2$ and $V = \mathbb{R}$, taking $f(x_0, x_1) = x_0^2 \sin(x_1)$ we have

$$\nabla f(x_0, x_1) = \begin{bmatrix} 2x_0 \sin(x_1) & x_0^2 \cos(x_1) \end{bmatrix}, \tag{4}$$

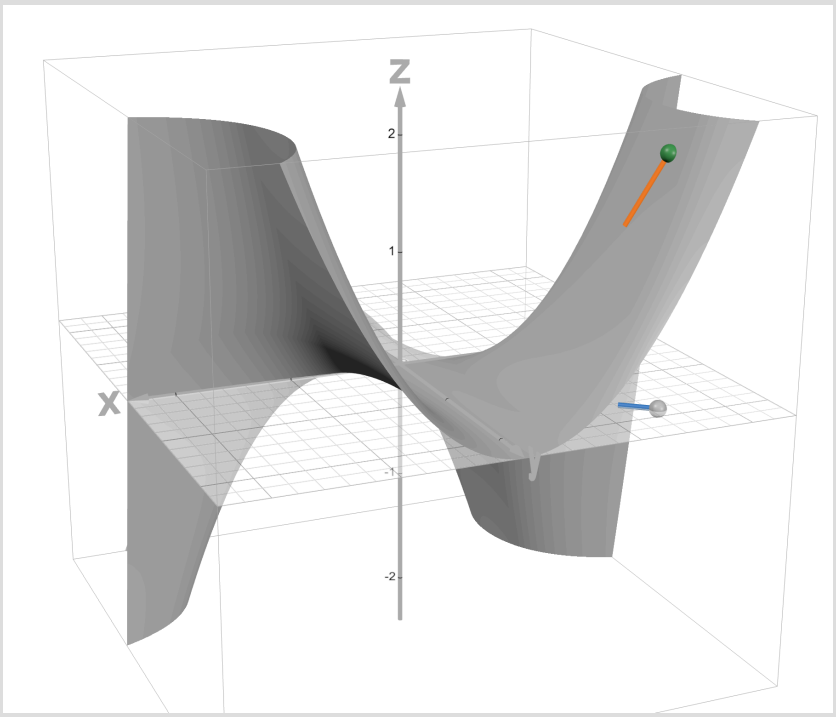
so $Tf : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ is the map

$$Tf : ((x_0, x_1), (v_0, v_1)) \mapsto (x_0^2 \sin(x_1), 2x_0 \sin(x_1)v_0 + x_0^2 \cos(x_1)v_1). \tag{5}$$

The functorial action of the tangent vector can be visualised as in the 3D diagram below, which you can access an interactive version on Desmos3D via the following URL: https://www.desmos.com/3d/kxc4zldyxz. Play around with the point p and vector v to get a feel for what the functorial action

[2] spoilers: its the derivative.

does, or even change the function f. Here is a picture I took from the demo, in case you cannot access it or are too lazy ((p, v) is the blue line segment, $Tf(p, v)$ is the orange segment):



2.2 The Tangent Functor and Projection Map

Using this motivating example, we now define tangent categories as follows. There's quite a lot of data and conditions, so the definition will be spread into some blocks.

Definition 2.2.1. A *tangent category* $(\underline{X}, T, p, 0, +, c, l)$ is a category $\underline{X}$ equipped with an endofunctor $T : \underline{X} \to \underline{X}$ and five natural transformations $p, 0, +, c, l$. The first of these is called the *projection*

$$p : T \Rightarrow 1_{\underline{X}}, \tag{6}$$

and we further require that all pullbacks along $T^k p_M$ exist for all $k \in \mathbb{N}$ and $M \in \mathrm{obj}(\underline{X})$, and that T^n preserves these pullbacks for all $n \in \mathbb{N}$, i.e. we ask for the following pullbacks for all n and k and M.

$$\begin{array}{ccc} T^n P & \xrightarrow{T^n \pi_1} & T^{n+k+1} M \\ \downarrow{\scriptstyle T^n \pi_0} & \times & \downarrow{\scriptstyle T^n(T^k(p_M))} \\ T^n N & \xrightarrow[T^n f]{} & T^{n+k} M \end{array}$$

(to be continued)

Obviously, $\underline{X}$ abstractly plays the role of $\underline{\mathsf{Smooth}}$ and T plays the role of the tangent functor. Note that we asked for certain pullbacks to exist, but not for all of them because e.g. $\underline{\mathsf{smooth}}$ doesn't have all pullbacks, here's an example.

Example 2.2.2. (Non-existent Pullbacks in $\underline{\mathsf{Smooth}}$) The pullback of the diagram below ought to be the cross $\{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ but as we have established this is not a manifold (and definitely not an open subset of $\mathbb{R}^2$), so this diagram has no pullback.

$$\begin{array}{ccc} & & \mathbb{R} \times \mathbb{R} \\ & & \downarrow{\scriptstyle (x,y) \mapsto xy} \\ 1 & \longrightarrow & \mathbb{R} \\ & {\scriptstyle \bullet \mapsto 0} & \end{array}$$

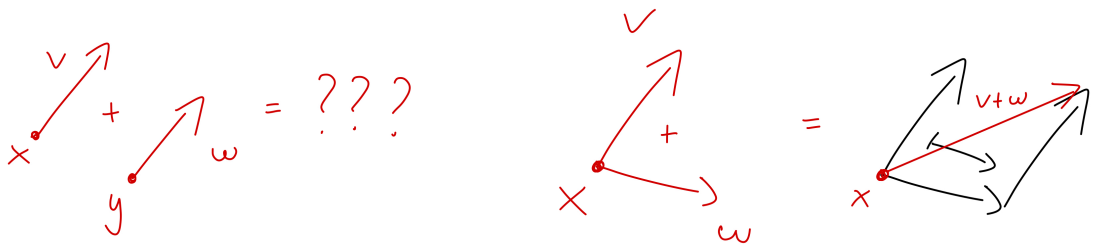
Example 2.2.3. (Pullbacks in $\underline{\mathsf{Smooth}}$ that are not preserved by T) The diagram on the left below has a pullback; it is the singleton manifold. However, the tangent functor does not preserve this pullback, because the tangent bundle of the singleton is still the singleton while the pullback of the tangent functor'ed diagram (as on the right below) is $\mathbb{R}$.

$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{R} \\ \downarrow & \times & \downarrow{\scriptstyle x \mapsto x^2} \\ 1 & \longrightarrow & \mathbb{R} \\ & {\scriptstyle \bullet \mapsto 0} & \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{v \mapsto (0,v)} & T\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \\ \downarrow & \times & \downarrow{\scriptstyle (x,v) \mapsto (x^2, 2xv)} \\ T1 \cong 1 & \longrightarrow & T\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \\ & {\scriptstyle \bullet \mapsto (0,0)} & \end{array}$$

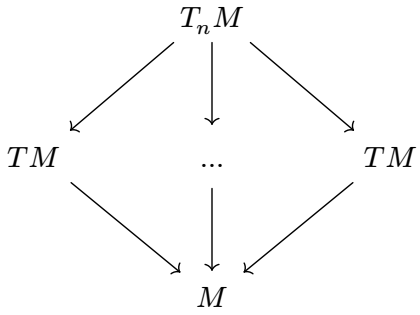
Here's how to make sense of this: the singleton point in the pullback of the lefthand diagram is the point $(0,0)$ in the graph of the quadratic function $x \mapsto x^2$ and the tangent space at this point is of course the line. So this point wants to have tangent bundle $\mathbb{R}$, but the definition of spaces in $\underline{\mathsf{Smooth}}$ is not "rich enough" to keep track of this desire, i.e. we cannot distinguish between a genuinely trivial point that wants to have tangent bundle 1 and a point that wants to have tangent bundle $\mathbb{R}$. We will see in Section 3 that it is possible to address precisely this issue in a different tangent category, the tangent category of schemes. So that's one motivation for tangent categories, i.e. to avoid reworking the machinery of differential geometry from the ground up when we invariably move to nicer categorical settings.

2.3 Tangent Vector Addition

The next two natural transformations concern the ability to add two tangent vectors together, and the identity/zero vector of this addition operation. Now, it doesn't make sense to be able to add any two vectors, but it makes sense when the two tangent vectors come out of the same point.



To make this precise we need to talk about the *pullback powers* $T_n M$ of n copies of p_M, i.e. the vertex of the limiting cone for the following diagram.



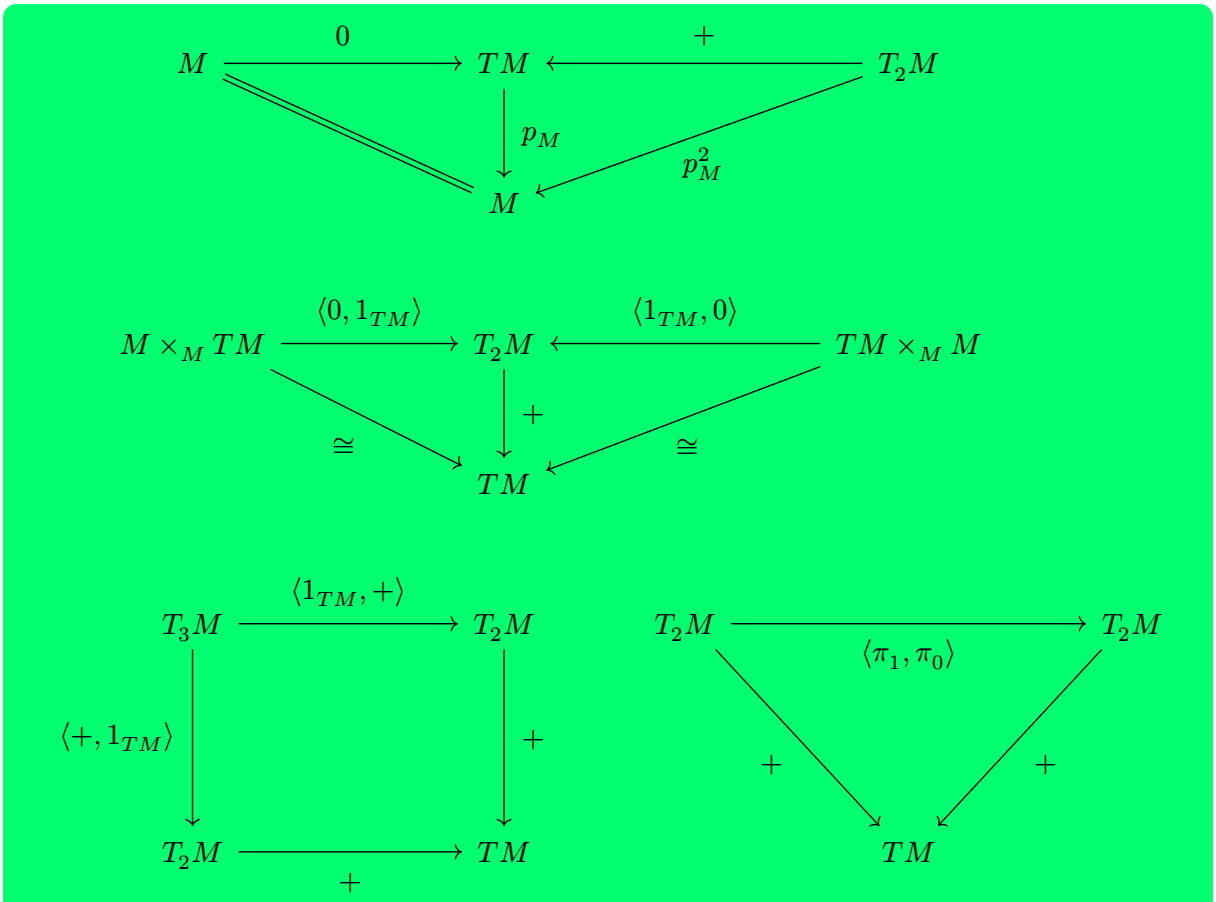
We will write $p_M^n : T_n M \to M$ for the map into M that this determines.

Example 2.3.1. In $\underline{\mathsf{Smooth}}$ an element of the pullback power is just n-many tangent vectors anchored at the same point x, and $p^n M$ sends such a family of vectors to x.

Definition 2.3.2. The next two natural transformations are (unsurprisingly) called the *zero* and the *plus*

$$0 : 1_{\underline{X}} \to T \quad + : T_2 \to 1_{\underline{X}}, \tag{7}$$

and these must, for each $M \in \mathrm{obj}(\underline{X})$, endow p_M with the structure of a commutative monoid in the slice category $\underline{X}/M$, i.e. making the following diagrams commute:



(to be continued)

Remark. Some natural questions here are why we don't make this an abelian group object or vector space for some ambient field object. The reason is that these limit the examples we can consider, for example there is *tropical geometry* where we work in settings without a negative operation. We will tack on the negative as an extra condition when we need it later in discussing the Lie bracket.

Example 2.3.3. These two natural transformations in the tangent category Smooth are defined as

$$0_M : M \to TM : x \mapsto (x, 0) \quad \text{and} \quad +_M : T_2M \to TM : (x, v, w) \mapsto (x, v + w). \tag{8}$$

2.4 Higher-order Tangent Vectors

The next natural transformation c pertains to second-order tangent vectors, i.e. those living in T^2M. There are two ways of projecting $T^2M \to TM$, the map c_M tells us how to go from one to the other.

Example 2.4.1. Let $U_n \in \text{obj}(\text{Smooth})$. Then

$$T^2U_n = T(U_n \times \mathbb{R}^n) = (U_n \times \mathbb{R}^n) \times \mathbb{R}^{2n} \cong (U_n \times \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \tag{9}$$

So an element of this second-order tangent bundle is a 4-tuple (x, u, v, a), where x is a point (zero-order tangent), u, v are first-order tangents, and a is a second-order tangent (the tangent vector at u). Here's a concrete example taking $U_n = \mathbb{R}$, so that $n = 1$. We think of this as a 1-dimensional physical space, so that a point x denotes a point x metres from the origin. We think of u and v as velocity vectors at point x, while a is an acceleration vector modifying u.

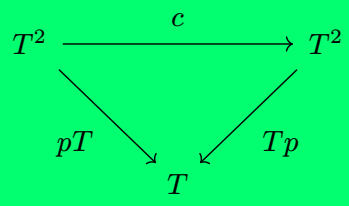
The two projection maps p_{TM} and $T(p_M)$, in this example, are

$$p_{TM} : (x, u, v, a) \mapsto (x, u) \quad \text{and} \quad T(p_M) : (x, u, v, a) \mapsto (x, v). \tag{10}$$

Definition 2.4.2. The fourth natural transformation is called the *canonical flip*:

$$c : T^2 \to T^2 \tag{11}$$

and it is required to satisfy $cc = 1_{T^2}$ and make the following diagram commute,



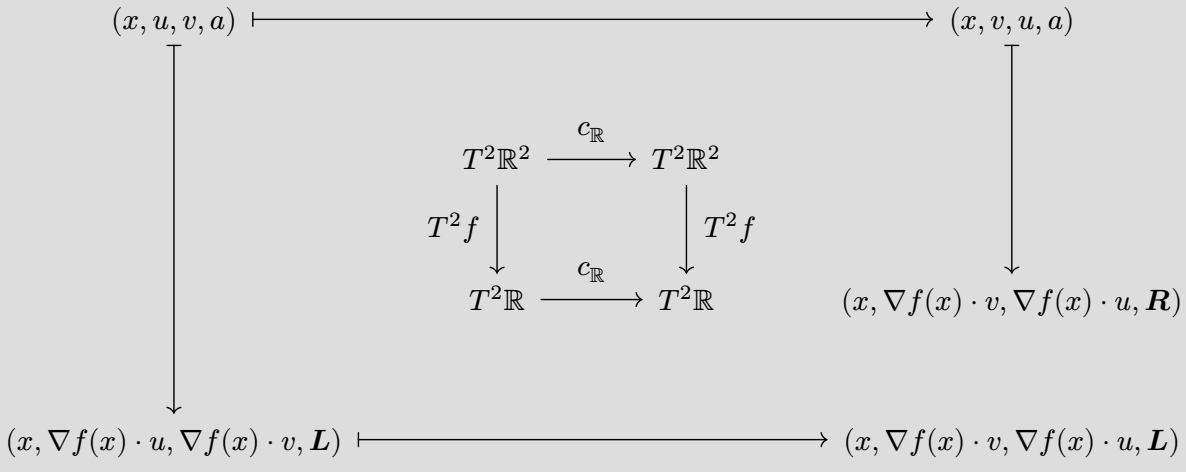
along with a *Yang-Baxter equation* which was glossed over in the lecture. Note that we can also derive the other triangle $c(pT) = ccTp = Tp$ from the two axioms above, so that c is an isomorphism between pT and Tp in the slice category over T.

(to be continued)

Example 2.4.3. In Smooth the canonical flip exchanges the two first-order tangent vectors in T^2:

$$c_M : (x, u, v, a) \mapsto (x, v, u, a). \tag{12}$$

It is instructive to prove why this map is natural in M, which for concreteness let's consider the following naturality square for some $f : \mathbb{R}^2 \to \mathbb{R}$.



where the second-order terms differ in the highlighted bit:

$$\boldsymbol{L} = \frac{\partial^2 f}{\partial x_0^2} u_0 v_0 + \frac{\partial^2 f}{\partial x_0 \partial x_1} u_1 v_0 + \frac{\partial^2 f}{\partial x_1 \partial x_0} u_0 v_1 + \frac{\partial^2 f}{\partial x_1^2} u_1 v_1 + \frac{\partial f}{\partial x_0} a_0 + \frac{\partial f}{\partial x_1} a_1 \tag{13}$$

$$\boldsymbol{R} = \frac{\partial^2 f}{\partial x_0^2} v_0 u_0 + \frac{\partial^2 f}{\partial x_0 \partial x_1} v_1 u_0 + \frac{\partial^2 f}{\partial x_1 \partial x_0} v_0 u_1 + \frac{\partial^2 f}{\partial x_1^2} v_1 u_1 + \frac{\partial f}{\partial x_0} a_0 + \frac{\partial f}{\partial x_1} a_1 \tag{14}$$

but the equality of the highlighted bit comes down to the well-known identity for partial derivatives which says it doesn't matter what order we take second-order derivatives in:

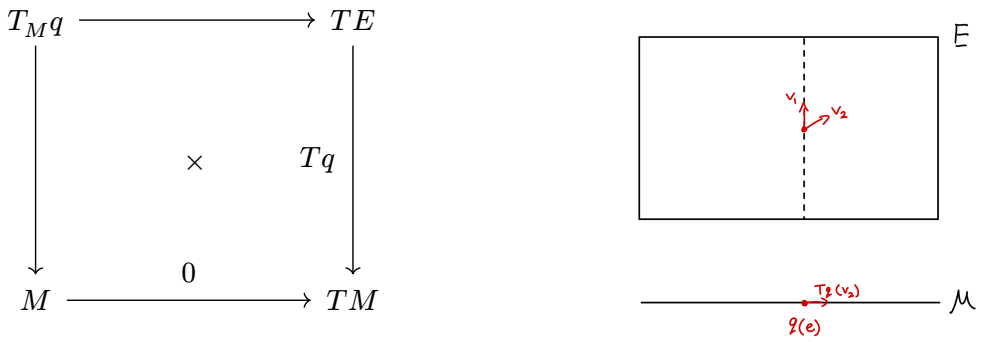
$$\frac{\partial f}{\partial x_0 \partial x_1} = \frac{\partial f}{\partial x_1 \partial x_0} \tag{15}$$

2.5 The "Bundle"-ness of $p_M : TM \to M$; the universality of the lift

Now, the final natural transformation and its associated condition is the hardest to make sense of, but also the most important one for grokking tangent categories. It is called the *lift* and is about making sure the tangent bundle $p_M : TM \to M$ is actually a "vector bundle", i.e. which "locally" looks like a "vector space" over each point of M. Roughly speaking, a "vector space" is an object V such that there is an isomorphism $\mu : V \times V \xrightarrow{\cong} TV$, i.e. where the tangent vectors and points of V amount to the same thing.

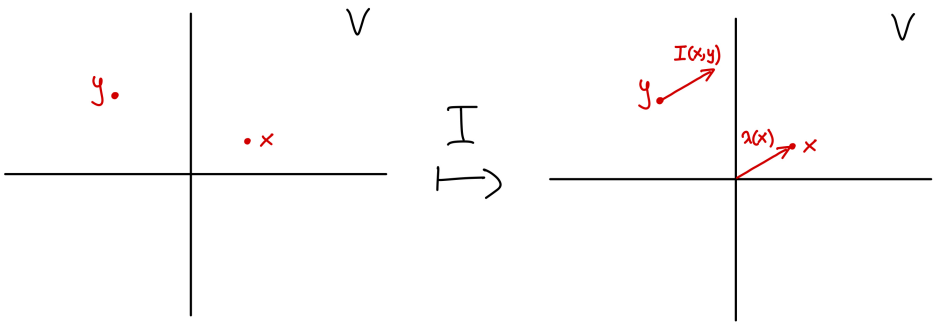
Example 2.5.1. In Smooth these are precisely the objects of the form $\mathbb{R}^n$ which are precisely the (finite-dimensional) vector spaces over the field $\mathbb{R}$.

Because we want p_M to be *locally* a "vector space", we need to make sure p_M is a differential object in the *slice* $\underline{X}/M$ (rather than in $\underline{X}$). This means that we want the product of p_M with itself in the slice, which is just the pullback power $p_M^2 : T_2M \to M$, to be isomorphic to the tangent object of p_M as an object in the slice. But what ought to be the tangent functor of a slice object $q : E \to M$? The answer is to take $T_M q$ the space of all those tangent vectors of E which project to the 0-vector in M. Via the following picture on the righthand side we can think of these as the "vertical" tangent vectors (such as v_1 in the picture, as opposed to v_2).



Remark. You might ask why this is the correct notion of tangent functor? Indeed, in the original paper [2] this is not the tangent functor that you take, and instead you take $T(q : E \to M) := p_E q$. This works, but isn't very interesting because it doesn't retain any "differential/tangential" information - it immediately projects away from TE, so no tangent vectors from E remain. The one we do take here does retain tangent vectors, namely the vertical ones. Why is this good? Well, in the slice category we're really focusing on things that happen "above M". And from the point of view of the picture, the vertical tangent vectors are really the ones that stay "in the fibre over M" - any ones which don't do this are in some sense taking into account "motion in M", but since we're focusing on "the fibres over M", we just want tangent vectors "which stay in the fibre over M". So that's why this is probably the "best" tangent bundle for a map over M.

However, this still isn't quite the proper definition of "vector space", because in fact V should also be a commutative monoid $(V, \sigma : V \times V \to V, \zeta : 1 \to V)$ and we want the isomorphism $\mu : V \times V \xrightarrow{\cong} TV$ to be linear on each argument. The explanation is that the isomorphism μ takes a pair (x, y) of points x and y and performs two steps: (1) it treats x as a vector $\lambda(x)$ from the origin pointing to x, and (2) it then shifts $\lambda(x)$ to a vector at y.

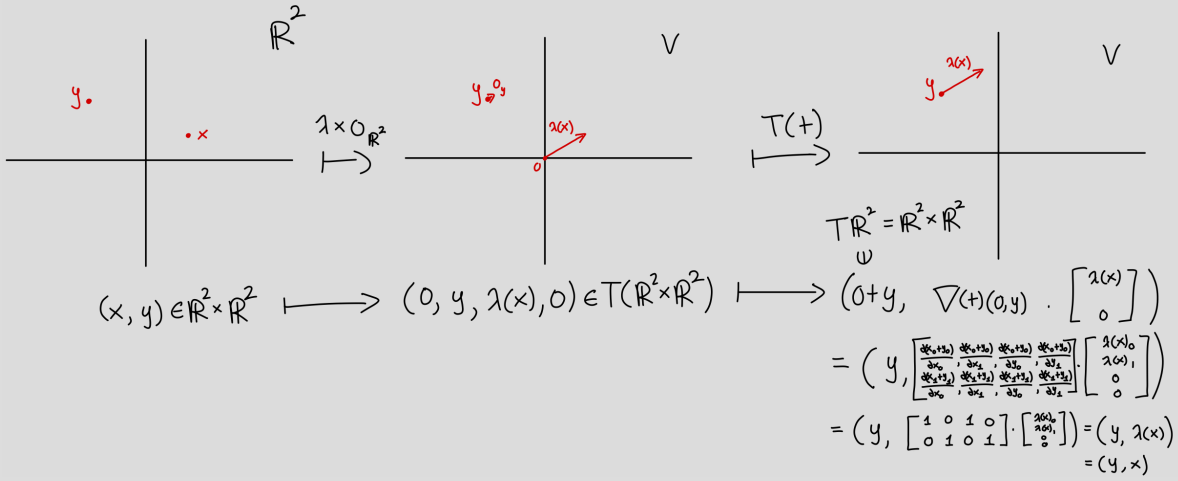


The first step yields a map

$$\lambda : V \to TV, \tag{16}$$

and the second step constructs the isomorphism μ by "regenerating" the second argument under addition: $\mu = V \times V \overset{\lambda \times 0_V}{\to} TV \times TV \cong T(V \times V) \overset{T\sigma}{\to} TV$ and it is this map that we demand to be an isomorphism when asking for V to be a "vector space". Note that here we used Definition 2.2.1 to have $TV \times TV \cong T(V \times V)$.

Example 2.5.2. Let's work with $V = \mathbb{R}^2$ in Smooth. Then the lift is $\lambda : x \mapsto (0, x)$, and we can compute the map μ, and see that it is indeed an isomorphism:

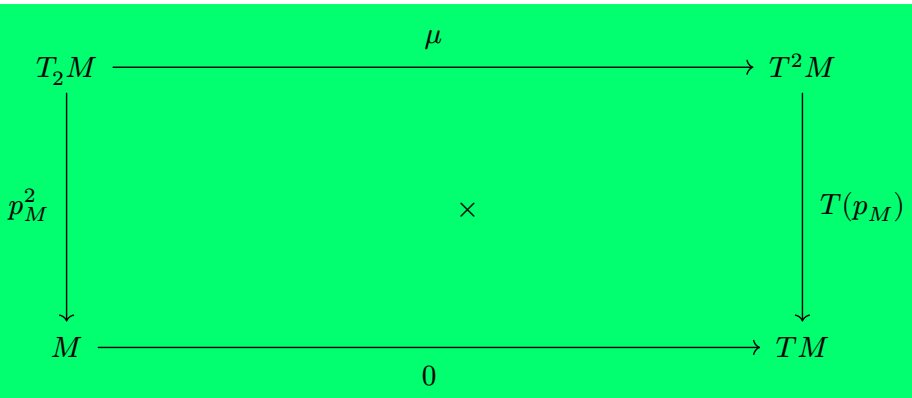


In Section 5 we formally define the "vector spaces" and "vector bundles" in exactly this vein, which for tangent categories are called the *differential objects* and *differential bundles* respectively. But basically the following condition amounts to asking that $p_M : TM \to M$ be a differential bundle, i.e. a differential object $V = p_M$ in the slice $\underline{X}/M$. The vertical lift $l_M : TM \to T^2M$ in the following definition is precisely the map $\lambda : V \to TV$ above re-interpreted in the slice, and the condition that the induced map $\mu : V \times V \to TV$ be iso is re-rexpressed as demanding that $V \times V = T_2M$ satisfy the pullback universal property defining $TV = T_M(p_M)$.

Definition 2.5.3. The final natural transformation is called the *vertical lift* or just *lift* for short:

$$l : T \to T^2 \tag{17}$$

and it is required to satisfy the *universality of vertical lift*, i.e. the following diagram has to be a pullback:



$$\text{where } \mu := T_2M \xrightarrow{l_M \times 0_{TM}} T^2M \times_{TM} T^2M \cong T(T_2M) \xrightarrow{T(+)} T^2M.$$

Moreover, the lift has to satisfy some coherence conditions with respect to the canonical flip, which was completely glossed over in the lectures. This finishes the definition of *tangent categories*.

Example 2.5.4. In Smooth the vertical lift is the map

$$l_M : TM \to T^2M : (x, v) \mapsto (x, 0, 0, v) \tag{18}$$

This is like turning a velocity vector v m/s into an acceleration vector v m/s^2. Since the physical intuition is kinda cursed, and in differential geometry textbooks this operation is present but not very prominent, its interesting that they become so important in the abstraction.

The map μ in this case is defined by

$$\mu : T_2M \to T^2M : (x, v, w) \mapsto (x, v, 0, w). \tag{19}$$

2.6 A Degenerate Example of Tangent Categories

Here's a degenerate example of a tangent category: any category $\underline{X}$ is a tangent category when equipped with $T = 1_{\underline{X}}$ and the obvious natural transformations. But these are really boring, they're basically saying that the objects have no non-zero tangent vectors and so are very "discrete" objects.

In the next two sections we will explore two more non-degenerate examples. The first of these concern some problematic spaces like the cross which arise as the space of zeroes for some polynomials. It turns out that while these are not manifolds, we can still meaningfully put a tangent bundle over them.

Remark. Before we move on, some remarks about how tangent categories are useful in settings "larger" than the smooth manifolds. One of the problems of the category of smooth manifolds is that it does not have exponentials. So people have come up with categories that embed the category of smooth manifolds but is also cartesian closed. Now, these have two candidate notions of tangent bundle called the "kinematic" and "operational" bundles, both of which restrict to the same thing on the smooth manifolds. But it turns out only the "kinematic" bundle supports a tangent category structure, so this helps resolve the question of which notion of bundle to prioritise. Other settings "larger" than smooth manifolds involve the diffeological spaces as well as the micro-linear objects of synthetic differential geometry (SDG), both of which form tangent categories.

Remark. A further remark on SDG. Category theory is to topos theory as tangent categories is to SDG. So the categories of SDG give you the nicest possible tangent categories, just as toposes are the

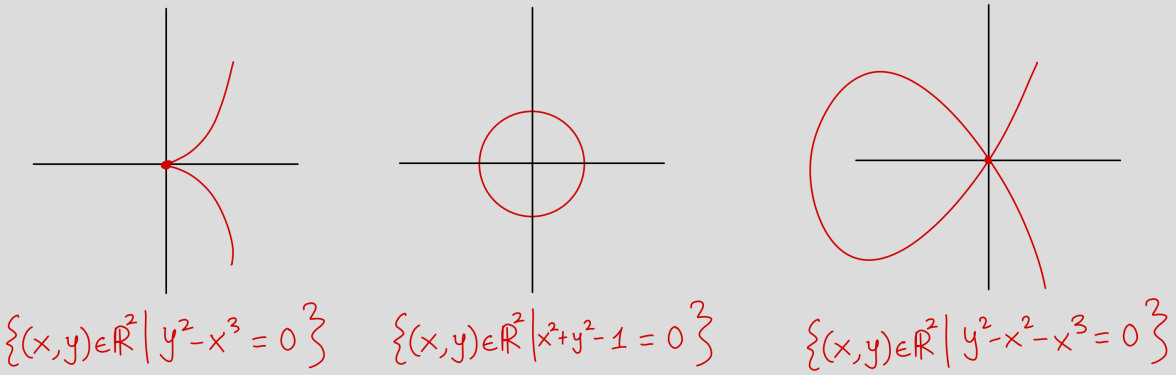
nicest possible categories. In particular the tangent categories of SDG has an object D classifying the tangent bundle, i.e. so that $TM \cong M^D$. This is not true for the smooth manifolds, for example.

3 Algebraic Geometry & Tangent Category of Affine Schemes

3.1 Affine Schemes

Look back at Example 2.2.3 of the single point which "wants to have a 1-dimensional tangent vector space". This happened because the point is really the zero of the quadratic equation x^2. Do you remember from high school algebra the fundamental theorem of algebra, which says that every complex polynomial of degree n has n roots? Except you had to do some bullsh*t like counting certain roots multiple times for the theorem to be true, such as the case with x^2. The study of the spaces of roots for polynomials is called "Algebraic Geometry", and these roots that "occur multiple times" lead to interesting phenomena, especially when one is concerned about tangent spaces. Another example we have already seen is the cross, which "wants to have a tangent space of dimension 2" at the center of the cross and "wants to have a tangent space of dimension 1" everywhere else. Here are some more examples.

Example 3.1.1.



How can we model such spaces? Well, the answer is we don't! Instead, we cheat by taking an algebraic structure and pretending as much as possible that they are spaces. The algebraic structure in question are *commutative rings*, or more generally *commutative algebras*[3] *over a base ring* R. This will lead us to define a tangent structure on the *opposite category* $\underline{\mathsf{Ring}}^{\mathrm{op}}$ or $\underline{\mathsf{cAlg}}_R^{\mathrm{op}}$. Having to work in the "opposite" is the price we pay for pretending! While we're pretending, we call an object $R \in \underline{\mathsf{cAlg}}_R^{\mathrm{op}}$ an *affine scheme* over R.

Suppose we had some family of polynomials $p_1 \ldots p_m$. We want to consider the object whose points are precisely the zeroes of $p_1 \ldots p_m$ (i.e. those points x so that $p_1(x) = \ldots = p_m(x) = 0$). Well, for that matter what is a point of an object? We know from category theory, generalising the category of sets, that a point ought to be a map $x : 1 \to A$. If we apply that definition to $\underline{\mathsf{cAlg}}_R^{\mathrm{op}}$, taking care of the op, we find that a point is an algebra homomorphism $x : A \to R$.

Example 3.1.2. As a sanity check, the points of of the algebra $\mathbb{R}^3$ of the ring $R = \mathbb{R}$ under scalar multiplication with element-wise product and addition consist of the algebra homomorphisms $x : \mathbb{R}^3 \to \mathbb{R}$. Since its an algebra homomorphism, we must have that $x(a, b, c) = x(a, 0, 0) +$

[3] these are the (commutative) monoids in the category of modules over R under the tensor product, fyi.

$x(0,b,c) + x(0,0,c)$. Moreover, each of those three values must, by preservation of scalar multiplication, be determined by $a \cdot x(1,0,0)$ and $b \cdot x(0,1,0)$ and $c \cdot x(0,0,1)$. So x is determined by three real numbers which we might as well call x_0, x_1, x_2, and any three such numbers determines an x. So the points of $\mathbb{R}^3$ are as we would expect.

In general, given a set X, there is the free algebra on X generators whose points are exactly R^X. With a bit of effort, we can work out that this free algebra is precisely the algebra of polynomials $R[X]$ with variables in X.

Now, in the category of sets, if we want to cut out a certain subset of points then we take an *equalizer*. So correspondingly, for affine schemes we want to take *coequalizers*, which are just quotients. Since the $p_1, ..., p_m$ can be understood as elements of $R[X]$, we can take the quotient $R[X]/_{\langle p_1 ... p_m \rangle}$ against the ideal generated by the elements $p_1 ... p_m$.

Example 3.1.3. The zeroes of $y^2 - x^2 - x^3$ is represented by the object $A = R[x,y]/_{\langle y^2 - x^2 - x^3 \rangle}$. By the universal property of the quotient, and of the free algebra, a point $p : A \to R$ corresponds to a function $\{x, y\} \to R$ which picks out two elements of R making the equation $y^2 - x^2 - x^3 = 0$ hold.

Now we can observe something interesting about the category of affine schemes. The (global) points of an object don't determine the object, the canonical example being $R[x]/_{\langle x^2 \rangle}$ versus R, where both have a single point. This solves the problem of Example 2.2.3, since the former algebra represents exactly the "point that wants to have a 1-dimensional tangent space"! In order to see this, let's define the tangent functor.

The tangent object TA for an algebra A can be defined by a universal property, which in particular implies T preserves all limits, not just pullbacks. But here we just give the explicit presentation.

Definition 3.1.4. Let A be an algebra. The *symmetric algebra of Kähler differentials* TA is the free algebra obtained by adjoining additional elements of the form da for each $a \in A$, subject to the relations

$$d1 = 0 \quad d(a+b) = da + db \quad d0 = 0 \qquad d(ab) = adb + bda \qquad \forall r \in R, a, b \in A \tag{20}$$

(Notice that these look exactly like how the derivative behaves.)

By construction, there is an algebra homomorphism

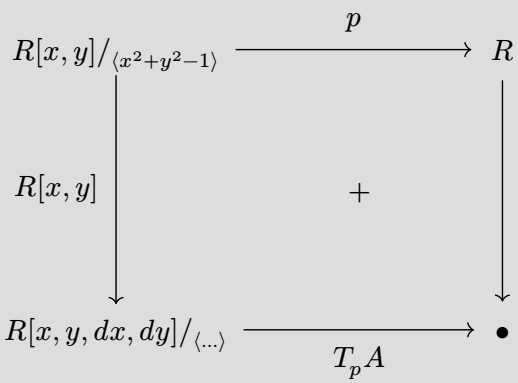
$$a \mapsto a : A \xrightarrow{p_A} TA \tag{21}$$

called the *projection* (can you guess which tangent category structure this is?)

Now we compute a couple examples.

Example 3.1.5. For $A = R[x]$, we end up finding that dp for any polynomial p can be expressed in terms of dx. For example, $d(x^2) = xdx + xdx = 2xdx$. Modulo being able to "divide" by the dx, this is exactly the formula for the derivative! So $TA \cong R[x, dx]$.

Example 3.1.6. For $A = R[x,y]/_{\langle x^2+y^2-1 \rangle}$, we simply have to add in the differentials dx and dy, and further quotient by $d(x^2 + y^2 - 1) = 0$. So we find that $TA \cong R[x, y, dx, dy]/_{\langle x^2+y^2-1, 2xdx+2ydy \rangle}$. What does this look like? Well we can check by "pulling back" over a particular local point $p : A \to R$ which of course becomes the following pushout of algebras:



Since p selects two values p_0 and p_1, this can be computed as

$$R[x,y,dx,dy]/_{\langle x^2+y^2-1,2xdx+2ydy,x-p_x,y-p_y\rangle} \cong R[dx,dy]/_{\langle 2p_0dx+2p_1dy\rangle} \tag{22}$$

But notice that the equation $2p_0dx + 2p_1dy$ exactly describes the tangent space at point p! See https://www.desmos.com/calculator/ofsuu4gmhf for an interactive example you can play with.

Example 3.1.7. For $A = R[x,y]/_{\langle y^2-x^2-x^3\rangle}$ we compute just as above that the tangent space at the "funky" point $p = (0,0)$ to be $T_pA = R[dx,dy]/_{\langle 2\cdot p_1\cdot dy-2p_0dx-3p_0^2dx}$. But since $p_0 = p_1 = 0$, the quotient does nothing and this reduces to $T_pA = R[dx,dy]$. So the tangent at this crossing point is a 2-dimensional vector space, precisely as we anticipated.

Example 3.1.8. Finally, for $A = R[x]/_{\langle x^2\rangle}$ we compute in much the same way that $T_0A = R[dx]$, and this is really a 1-dimensional space as anticipated!

What is the functorial action of the Kahler differential? It's not complicated at all. Given $f : A \to B$, we take

$$Tf : TA \to TB : \begin{cases} a \mapsto f(a) \\ da \mapsto df(a) \end{cases} \tag{23}$$

The commutative monoid structure is given by

$$0_A : TA \to A : \begin{cases} a \mapsto a \\ da \mapsto 0 \end{cases} \qquad +_A : TA \to T_2A \cong TA \otimes_A TA : \begin{cases} a \mapsto a \\ da \mapsto 1 \otimes da + da \otimes 1 \end{cases} \tag{24}$$

For the canonical flip and the lift, first observe that T^2A has generating elements of the form a, da, $d'a$ or $d'da$. Then the flip and lift look like the "reindexings" performed by the corresponding flip and lift in Smooth:

$$l_A : T^2A \to TA : \begin{cases} a \mapsto a \\ da \mapsto 0 \\ d'a \mapsto 0 \\ d'da \mapsto da \end{cases} \qquad c_A : T^2A \to T^2A : \begin{cases} a \mapsto a \\ da \mapsto d'a \\ d'a \mapsto da \\ d'da \mapsto d'da \end{cases} \tag{25}$$

Remark. Question from Alyssa: If M is a smooth manifold, how does TM relate to $TC^\infty(M,\mathbb{R})$, where $C^\infty(M,\mathbb{R})$ is the ring of smooth functions $M \to \mathbb{R}$?

Remark. We can extend this tangent category structure for schemes in general, roughly by gluing the TA for each local copy of A making up a scheme. In fact, starting from a tangent category $\underline{\mathbb{X}}$ there is a general theory [2] of obtaining a tangent category whose objects look like manifolds, i.e. glued copies of objects from $\underline{\mathbb{X}}$, provided the tangent category structure in $\underline{\mathbb{X}}$ extends to partial maps and thus exhibiting the structure of a *join-restriction tangent category*.

4 Tangent Category of Rings and Dual Numbers

The category $\underline{\mathsf{cAlg}}_R$ (no op!) has a tangent category structure, given by

$$TA = A(\epsilon) = A[\epsilon]/_{\langle \epsilon^2 \rangle} = \{a + b\epsilon \mid \epsilon^2 = 0, a, b \in A\} \tag{26}$$

This example is interesting, because it is a tangent category structure on a category whose objects don't really have much geometric intuition at all. Also, most of the theory of tangent categories developed via the geometric examples are trivial or useless in this tangent category.

TODO
An exercise in the lectures is to work out the tangent category structure maps here.

5 Differential Objects & Bundles

5.1 Differential Objects

We have already explained in-depth in Section 2 how one might define the notion of vector space in a tangent category. So here it is:

Definition 5.1.1. A *differential object* $(A, \sigma, \zeta, \lambda)$ in a tangent category $(\underline{\mathbb{X}}, T)$ is a commutative monoid object $(A, \sigma : A \times A \to A, \zeta : 1 \to A)$ together with a map $\lambda : A \to TA$ which "sends a point $a \in A$ to the vector $\lambda(a)$ coming from the origin", in such a way that the map

$$A \times A \overset{\lambda \times 0_A}{\to} T(A \times A) \overset{T\sigma}{\to} TA \tag{27}$$

is an isomorphism. This is required to satisfy some coherence axioms which we glossed over in the lectures.

Moreover, a *linear map* between two differential objects is just a monoid map which preserves the λ.

Example 5.1.2.

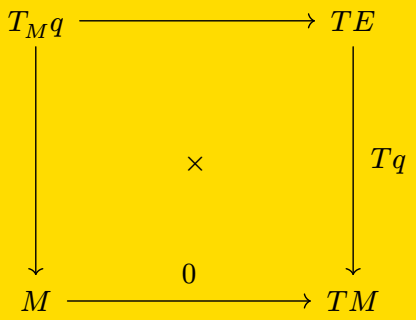
1. In the tangent category of Smooth manifolds, the differential objects are precisely the $\mathbb{R}^n$ for any n.
2. In the tangent category of affine schemes over R the differential objects and linear maps form a category equivalent to the opposite category of right R-modules.

Remark. (Relationship with Cartesian Differential Categories) If $f : A \to B$ is a not-necessarily-linear map between differential objects, we get a map $A \times A \cong TA \overset{Tf}{\to} TB \cong B \times B$ from which we can extract the maps $Tf\pi_0 : A \times A \to B$ and $Tf\pi_1 : A \times A \to B$. The first map is guaranteed to satisfy $Tf\pi_0 = \pi_0 f$, but the second map $Tf\pi_1$ gives us the derivative of f, and it satisfies various properties which give rise to an instance of cartesian differential categories.

5.2 Differential Bundles

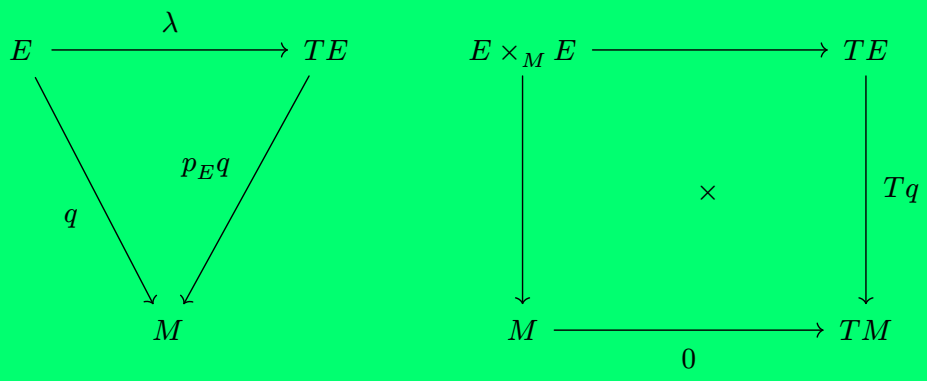
As we have already observed multiple times in Section 2, to get notions that apply to bundles we simply have to work in the slice category. Therefore it is good to know that the slice of a tangent category is again a tangent category, and this is fortunately the case.

Proposition 5.2.1. If $(\underline{\mathbb{X}}, T)$ is a tangent category then the slice category $\underline{\mathbb{X}}/M$ is again a tangent category with tangent functor $T_{M(q:E\to M)}$ given by the pullback



Again, we think of this as capturing the *vertical* tangent vectors in E.

Definition 5.2.2. A *differential bundle* over M is a differential object in $\underline{\mathbb{X}}/M$. Explicitly, it is a tuple $(q, \sigma, \zeta, \lambda)$ where $q : E \to M$ admits a commutative monoid structure $\sigma : E \times_M E \to E$ and $\zeta : M \to E$ in $\underline{\mathbb{X}}/M$ and the map $\lambda : E \to TE$ makes the diagram on the left commute and makes the diagram on the right a pullback.



As usual, there are additional coherence axioms that we disregard for brevity.

Example 5.2.3.

1. For smooth manifolds, every smooth vector bundle (as defined in the usual way) is a differential bundle. Whether the converse is true depends on the definition; some definitions require smooth vector bundles to have a single vector space V which the bundle looks similar to everywhere (a uniformity condition). In general, a differential bundle need not look like any single vector space V everywhere, so if this additional uniformity condition is dropped, then we get an equivalence (See Theorem 4.27 of [13]).
2. In affine schemes, the differential bundles are the opposite category of modules over some algebra A. In the full tangent category of schemes, the differential bundles are the category of quasi-coherent sheaves of modules.
3. In $\underline{\mathsf{cRing}}$, the differential bundles over a ring R are precisely the modules over R (no opposite!) In particular, however, the differential objects are the differential bundles over the terminal object;

in this category, the terminal object is the 0-ring, and the only modules over the 0-ring are the trivial modules, so there are no non-trivial differential objects. See [8] for more details.

6 Vector Fields and the Lie Bracket

6.1 Vector Fields and their Algebraic Operations

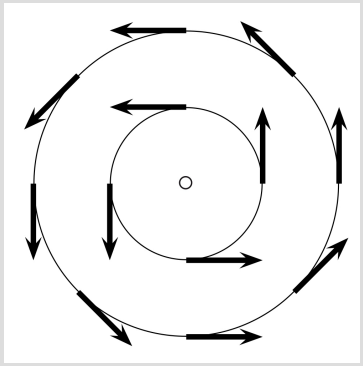
A vector field is a choice of tangent vector at each point of M. These appear a lot in physics, for example to the describe the gravitational field generated by a massive object, or the magnetic field of a magnet. Categorically it is very easy to define vector fields.

Definition 6.1.1. A *vector field* on $M \in \underline{X}$ is a section of $p_M : TM \to M$, i.e. a map $v : M \to TM$ s.t. $vp_M = 1_M$.

Example 6.1.2. You've already seen an example: for any M, $0_M : M \to TM$ is a vector field. Here is a non-degenerate example in Smooth. We are defining a vector field v on $U_2 = \mathbb{R}^2 - \{(0,0)\}$. We define it to be (notice that the first component is forced to be (x, y))

$$v : (x, y) \mapsto \left((x, y), \left(-\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right) \right) \tag{28}$$

This defines a "vortex" vector field, the normalizing division makes sure that the vortex remains the same strength no matter how far we are from the center $(0, 0)$, as pictured below (taken from [14]).



We have the 0 vector field, and we can also add two vector fields together.

$$v + w : M \overset{\langle v,w \rangle}{\to} T_2 M \overset{+_M}{\to} TM \tag{29}$$

In fact, the set of vector fields itself forms a vector space. This vector space has a binary operation on it, called the *Lie bracket* AKA the *Lie derivative*, making it an $\mathbb{R}$-algebra (and not just any old algebra, but a *Lie algebra*!)

6.2 The Lie Bracket

Given a vector field v and a point p, we can think of v as moving us from p in the direction of $v(p)$ infinitesimally to some new point q. Given another vector field w we can then ask for how $w(q)$ varies along this infinitesimal, that is the "derivative of w along v":

$$M \overset{v}{\to} TM \overset{Tw}{\to} T^2 M \tag{30}$$

Of course we could also have reversed the roles of w and v, and get the "derivative of v along w":

$$M. \overset{w}{\to} TM \overset{Tv}{\to} T^2M \tag{31}$$

It is then natural to ask whether these two are the same. In general, the answer is no.

Example 6.2.1. Consider the vector field $v : \mathbb{R}^2 \to T\mathbb{R}^2 : (x, y) \mapsto ((x, y), (1, 0))$ and $w : \mathbb{R}^2 \to T\mathbb{R}^2 : (x, y) \mapsto ((x, y), (0, x))$. Then we can compute:

$$vTw : (x, y) \overset{v}{\mapsto} ((x, y), (1, 0)) \overset{Tw}{\mapsto} \left((x, y), (0, x), \nabla w(x, y) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \left((x, y), (0, x), \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial 0}{\partial x} & \frac{\partial 0}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left((x, y), (0, x), \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}\right), \left(\frac{\partial 0}{\partial x}, \frac{\partial x}{\partial x}\right)\right) \tag{32}$$

$$= ((x, y), (0, x), (1, 0), (0, 1))$$

whereas for the other derivative,

$$wTv : (x, y) \overset{w}{\mapsto} ((x, y), (0, x)) \overset{Tv}{\mapsto} \left((x, y), (1, 0), \nabla v(x, y) \cdot \begin{pmatrix} 0 \\ x \end{pmatrix}\right)$$

$$= \left((x, y), (1, 0), \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial 1}{\partial x} & \frac{\partial 1}{\partial y} \\ \frac{\partial 0}{\partial x} & \frac{\partial 0}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x \end{pmatrix}\right) = \left((x, y), (1, 0), \left(x\frac{\partial x}{\partial y}, x\frac{\partial y}{\partial y}\right), \left(x\frac{\partial 1}{\partial y}, x\frac{\partial 0}{\partial y}\right)\right) \tag{33}$$

$$= ((x, y), (1, 0), (0, x), (0, 0))$$

Note that in general, the first-order vectors will just be the result of v and w in one of two possible orders. So the real difference is in the second-order component, as highlighted, which actually measures the derivative of one vector field against the other. Here we see that of course the derivative of v against w is $(0, 0)$, since v is a constant map. On the other hand the derivative of w against v is non-trivially $(0, 1)$ since w actually depends on the input.

We would like a way of measuring the difference between these two ways of taking the derivatives, and in fact we would like to think of this difference measure as a kind of "unbiased derivative" in itself.

First of all then, we had better ensure our tangent bundles have negatives, i.e. is an abelian group rather than just a commutative monoid.

Definition 6.2.2. A *tangent category with negatives* $(\underline{X}, T)$ is a tangent category $(\underline{X}, T)$ further equipped with a natural transformation $- : T \to T$ which together with $+$ and 0 endows each $p_M : TM \to M$ with the structure of an abelian group object in $\underline{X}/M$.

Example 6.2.3. All our examples so far have negatives. The negative in $\underline{\mathsf{Smooth}}$ is

$$- : U_n \times \mathbb{R}^n \to U_n \times \mathbb{R}^n : (p, v) \mapsto (p, -v) \tag{34}$$

while in $\underline{\mathsf{cAlg}}_R^{\mathsf{op}}$ it is

$$- : TA \to TA : \begin{cases} a \mapsto -a \\ da \mapsto -da \end{cases} \tag{35}$$

Now, as we saw in Example 6.2.1, the vTw and wTv differ in the order in which the first-order tangent vectors appear, i.e. vTw is a slice map $w \to p_{TM}$ whereas wTv is a slice map $w \to T(p_M)$. But we know exactly how to go from p_{TM} to $T(p_M)$, its precisely the canonical flip c! So we can take the difference as follows:

$$vTw - wTv : M \xrightarrow{\langle vTw, wTvc_M -_{TM}\rangle} T_2TM \overset{+_{TM}}{\to} T^2M, \tag{36}$$

where the two components of the first map are

$$vTw : M \overset{v}{\to} TM \overset{Tw}{\to} T^2M \quad \text{and} \quad wTvc_M - : M \overset{w}{\to} TM \overset{Tv}{\to} T^2M \overset{c_M}{\to} T^2M \overset{-_{TM}}{\longrightarrow} T^2M \tag{37}$$

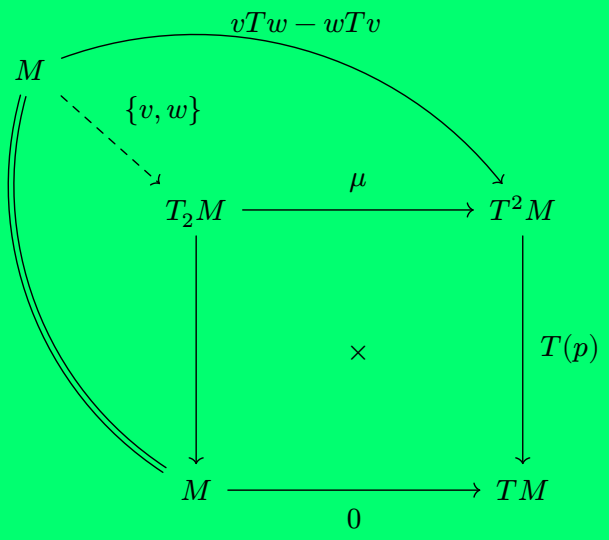
Example 6.2.4. We continue the story of Example 6.2.1. The difference of vTw and wTv is

$$(x, y) \mapsto ((x, y), (0, 0), (0, 0), (0, 1)). \tag{38}$$

Notice that since we subtracted them out, the first-order components are always 0.

In general, the first-order components are always 0, and so we can "unlift" the second-order vector into a first-order vector, i.e. we have an induced map into the pullback of Definition 2.5.3:

Definition 6.2.5.



and if we take the second projection

$$[v, w] := M \xrightarrow{\{\{v,w\}\}} T_2M \overset{\pi_1}{\to} TM \tag{39}$$

we get the unlifted version of the difference, another vector field! This is called the *Lie bracket* $[v, w]$, or the *Lie derivative of w along v*.

Remark. This binary operation on vector fields behaves well with the vectorial operations on vector fields, but it is not commutative. Rather, it is *anticommutative* since $[v, w] = -[w, v]$. Also, it is not associative but rather satisfies what is called the *Jacobi identity*:

$$[u,[v,w]] = [w,[u,v]] + [v,[w,u]]. \tag{40}$$

Together these properties make the vector space of vector fields a *Lie algebra*. Proving this categorically is very gnarly, and Geoffrey Cruttwell and Robin Cockett finally did it using a graphical calculus [15], and it still took 20 pages!

6.3 Commutative Vector Fields

Definition 6.3.1. In the case that $[v,w] = 0$, we say that the two vector fields commute with each other.

In fact, there is a way of arriving at the concept of commutative vector fields, purely by tangent categorical considerations. More precisely, what if we wanted a tangent category whose objects are vector fields?

Definition 6.3.2. Given a tangent category $(\underline{\mathbb{X}}, T)$ we define a new category $\underline{\mathsf{Vf}}(\underline{\mathbb{X}}, T)$ whose objects are pairs (M, v) of objects $M \in \mathrm{obj}(\underline{\mathbb{X}})$ and v a vector field on M, and whose morphisms $f : (M, v) \to (M', v')$ are maps $f : M \to M'$ making $(Tf)v = v'f$ commute.

Certainly, given (M, v) we can get (TM, Tv) which has the correct type, so a good first candidate for our tangent functor. But we still have to check that Tv is a vector field. Alas, we find that $(Tv)p_{TM} = p_M v \neq 1_M$. But of course, we know how to solve this: canonical flip! We take $(Tv)c_M$ instead, since

$$(Tv)c_M p_{TM} = TvT(p_M) = T(vp_M) = T(1_M) = 1_{TM}. \tag{41}$$

Theorem 6.3.3. $\underline{\mathsf{Vf}}(\underline{\mathbb{X}}, T)$ is a tangent category with $T(M, v) := (TM, (Tv)c_M)$.

And now for the magic trick: what is a vector field in $\underline{\mathsf{Vf}}(\underline{\mathbb{X}}, T)$? Well, let's unpack the definition: it is a map $w : (M, v) \to (TM, (Tv)c_M)$ so that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\quad w \quad} & TM \\ \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle (Tv)c_M} \\ TM & \xrightarrow{\quad Tw \quad} & T^2M \end{array}$$

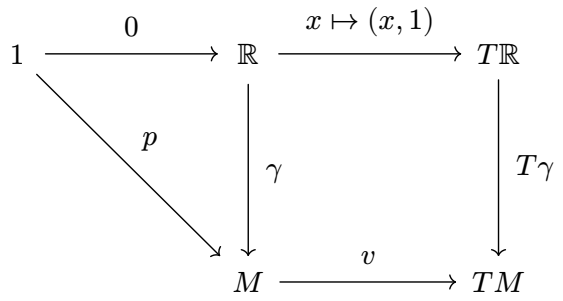
But notice that the two sides of the diagram correspond to the two things we take the difference of in the Lie bracket, so this diagram commutes precisely when $[v,w] = 0$.

7 Differential Equations

Suppose you had a vector field v describing the gravitational field of some space M, and you have a particle at a point $p : 1 \to M$. Then unless v is zero at p, p will start to move along the gravitational field. In other words, we end up with a map

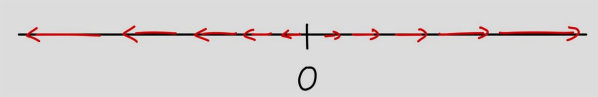
$$\gamma : \mathbb{R} \to M \quad \text{with} \quad \gamma(0) = p \tag{42}$$

, where $\gamma(t)$ describes the position of the particle at time t. Moreover, since γ is the movement of the particle according to v, its derivative at each point must be the same as what v describes. So we find that the following diagram ought to commute:



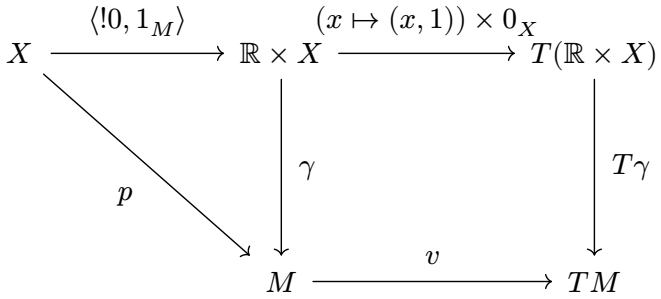
Now, in differential geometry, given M, v and p as above and finding the corresponding γ is a matter of solving some *ordinary differential equations*.

Example 7.0.1. Suppose $M = \mathbb{R}$ and $v : \mathbb{R} \to T\mathbb{R} : x \mapsto (x, x)$:



Then asking for the solution $\gamma : \mathbb{R} \to \mathbb{R}$ for a particle at position $p = 1$ is asking for $\gamma(0) = 1$ and $\gamma'(t) = \gamma(t)$. The solution is therefore $\gamma(t) := e^t$.

Of course, we can generalize this story from talking about global points $p : 1 \to M$ to just any generalized point $p : X \to M$, appropriately modifying the commutative diagram above:



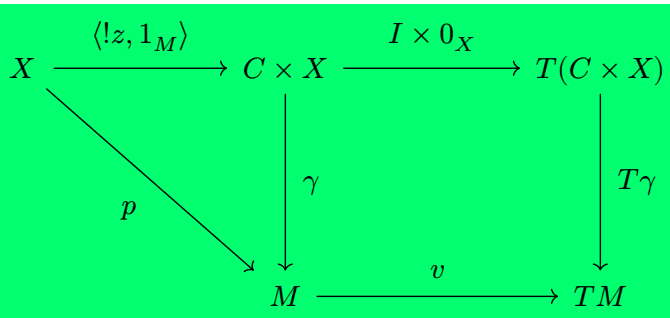
In particular, taking $p = 1_M$ and solving for $\gamma : \mathbb{R} \times M \to M$ yields what is called the *flow* of the vector field v.

Remark. Unfortunately, solutions/flows need not always exist. For example,

$$v : \mathbb{R} \to \mathbb{R} : x \mapsto (x, x^2) \tag{43}$$

needs a γ such that $\gamma'(t) = \gamma'(t) = \gamma(t)^2$, and the solution is $\gamma(t) = -\frac{1}{t-1}$ but this is not everywhere defined (particularly, at $t = 1$). One way to handle this is to work in the tangent category whose maps are partial maps, so that solutions are partial by definition. These are axiomatised by *tangent restriction categories*. But for this mini-course we will just stick to asking for uniqueness of certain solutions.

Definition 7.0.2. A *curve object* in a tangent category $(\underline{X}, T)$ is an object C together with a point $z : 1 \to C$ and a distinguished vector field $I : C \to TC$ such that "solutions relative to I are unique if they exist", i.e. for any $p : X \to M$ and $v : M \to TM$, if there is a γ making the following diagram commute,



then γ is the unique such map. Moreover, we demand that there is a solution σ (necessarily unique) for $p = 1_C$ and $v = I$, i.e. C solves its own flow problem.

Example 7.0.3. In Smooth, we take $C = \mathbb{R}$ and $I : x \mapsto (x, 1)$. If we solve for the flow then we find the map $\mathbb{R} \times \mathbb{R} \to \mathbb{R} : x, y \mapsto x + y$ which is exactly addition!

Indeed, any curve object forms a commutative monoid (C, z, σ). Naturally, you would ask if this is a differential object. This turns out not to be true in general, but you can ask for it, and also ask for solutions to "linear problems":

Definition 7.0.4. A *differential curve object* in a tangent category $(\underline{X}, T)$ is a curve object (C, z, I) such that (C, z, σ) is a differential object, and such that "every linear vector field has a solution".

With a differential curve object, and negatives, you get a *lot* of structure in your tangent category which makes it look much more like smooth manifolds, e.g. you get

1. trigonometric functions;
2. that C is an *internal ring object* in $\underline{X}$;
3. Every differential object and differential bundle acquires an action by C, i.e. become C-modules.

8 Connections

TODO
TODO

That's all folks!

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