Dependent two-sided fibrations for directed type theory

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Background

In a seminal paper, Hofmann and Streicher [1] constructed the *groupoid model* for Martin-Löf Type Theory (MLTT). In this model, a type A is interpreted as a groupoid $[\![A]\!]$, while its identity type Id_A is interpreted as the hom-set of $[\![A]\!]$. Further work has shown that types can be also interpreted as ∞-groupoids [2], and that, together with the univalence axiom, the resulting theory allows for a synthetic development of ∞-groupoids [3].

More recently, new, *directed*, type theories have emerged, with the goal of developing a way of doing synthetic category theory. Different approaches have been used, such as a 2-dimensional theories [4], modal typings [5, 6, 7], or multilayered approaches [8], among others. Arguably the most successful of these are Simplicial Type Theory [8] and Triangulated Type Theory [6]. However, they achieve their expressive power by interpreting types not as categories, but more general structures, and then carving out those types that behave like categories.

Our work aims to develop a type theory with comparable expressive power to the aforementioned ones, but without changing the universe of types. It builds on previous joint work with Mangel [9], which itself is a continuation of the work done in [10]. The key new contributions are a new context extension rule and modified rules for hom-types, which we now sketch.

An overview of directed type theory

We sketch some basic components in our theory, which were already present in [10]. Extending the groupoid model, we define contexts to be categories, with the empty context being the terminal category. A type in a context Γ is a functor $A : \Gamma \to \mathsf{Cat}$, it follows that a type in the empty context is a category. The context extension operation $\Gamma \triangleright A$ is interpreted as the Grothendieck construction $\int_\Gamma A$. As usual, terms of A are sections of the canonical projection $\Gamma \triangleright A \to \Gamma$.

Furthermore, we have types A^{core} and A^{op} for each type A, which in the semantics are obtained by postcomposing $A : \Gamma \to \mathsf{Cat}$ with the endofunctors $\mathsf{core}, \mathsf{op} : \mathsf{Cat} \to \mathsf{Cat}$, respectively. We additionally have a hom-type former, with formation rule:

$$\text{hom-}\textsc{Form}\ \frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma, x : A^{\mathsf{op}}, y : A \vdash \hom_A(x, y)\ \mathsf{type}}$$

In the empty context, this type is interpreted as the usual functor $\hom_A : A^{\mathsf{op}} \times A \to \mathsf{Set}$, except that each set is interpreted as a discrete category.

The dependent 2-sided context extension

In addition to the usual context extension rule, we now have a “dependent 2-sided” version:

$$\frac{\overset{\textsc{Ctx-Ext}_2}{\Gamma\ \mathsf{ctx} \qquad \Gamma \vdash A\ \mathsf{type} \qquad \Gamma, a : A^{\mathsf{op}} \vdash B(a)\ \mathsf{type}}}{\Gamma, a : A, b \overset{2\mathfrak{f}}{:} B(\bar{a})\ \mathsf{ctx}}$$

Semantically, it corresponds to a dependent analogue of Street's notion of a 2-sided fibration [11], which we call a **dependent 2-sided fibration (D2SFib)**. This construction is equipped with an opfibration over Γ and a "local fibration" over $\int_\Gamma A$, and is the universal such construction in the sense that we obtain an equivalence of categories

$$\mathsf{Functor}(\int_\Gamma (\mathsf{op} \circ A), \mathsf{Cat}) \simeq \mathsf{D2SFib}(\Gamma, A)$$

We now interpret terms $\Gamma, a : A \vdash b \overset{2\mathfrak{f}}{:} B$ as sections $(\Gamma, a : A) \to (\Gamma, a : A, b \overset{2\mathfrak{f}}{:} B)$.

However, D2SFibs are not stable under pullback, which requires us to make a new judgment to capture all substitutions. We write $\Gamma \vdash X\ \mathsf{type}_{\mathfrak{d}}$ for any displayed category $X \to \Gamma$; equivalently, for a normal lax functor X from Γ to the double category of profunctors Prof. Now, terms in X correspond to sections of the displayed category, and so we can write:

$$\frac{\overset{\textsc{Subst}}{\Gamma, x : X, \Delta \vdash a \overset{2\mathfrak{f}}{:} X \qquad \Gamma \vdash m : X}}{\Gamma, \Delta[m/x] \vdash a \overset{\mathfrak{d}}{:} X}$$

New rules for the hom-type

In particular, when applying this new operation to the hom-type we are able to form the new context $b : A, \bar{a} : A, f \overset{2\mathfrak{f}}{:} \hom_A(a,b)\ \mathsf{ctx}$ (note the switching of the variables). Semantically, this corresponds to the arrow category $A^{\to}$, which validates the following rules:

$$\frac{\overset{\text{hom-}\textsc{Intro}}{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash x : A}}{\Gamma \vdash \mathsf{refl}_x \overset{\mathfrak{d}}{:} \hom(\bar{x}, x)} \qquad \frac{\overset{\text{hom-}\textsc{Elim}}{\Gamma, a : A \vdash X\ \mathsf{type} \quad \Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(\bar{a}, b), x : X^{\mathsf{op}} \vdash D\ \mathsf{type} \quad \Gamma, a : A, x : X \vdash d \overset{\mathfrak{d}}{:} D[a/b, \mathsf{refl}_A/f]}}{\Gamma, b : A, a : A, f \overset{2\mathfrak{f}}{:} \hom_A(\bar{a}, b), x : X \vdash j_d \overset{2\mathfrak{f}}{:} D}$$

As an example of the semantics, the hom introduction rule in the empty context states that the diagonal $\Delta : A \to A \times A$ lifts along the projection $A^{\to} \to A$. For another example, we can now precisely capture natural transformations between two functors $F, G : A \to B$, they correspond to judgements $a : A \vdash \tau_a : \hom(F\bar{a}, Ga)$.

Furthermore, these rules plus the directed analogue of function extensionality are sufficient to develop some category theory. Indeed, we can show:

Lemma (Yoneda)**.** *Let $A : \mathcal{U}^{\mathsf{core}}$ be a type. Then, the following two functors are naturally* *isomorphic*

$$
\begin{aligned}
Y, \bar{Y} &: (A \to \mathsf{Set}) \times A \to \mathsf{Set} \\
Y(F, a) &:\equiv \prod_{(x,f) : \sum_{x:A} \hom(a,x)} Fx \\
\bar{Y}(F, a) &:\equiv Fa
\end{aligned}
$$

Future works

This is still work in progress; in particular, we need to better understand D2SFibs, their pullbacks, and their relation with factorization systems. Additionally, we still have to investigate just how much category theory can be developed internally. We hope that properties of Triangulated Type Theory, such as a directed structure identity principle, have an analog for our syntax.

References

[1] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-Five Years of Constructive Type Theory*, pages 83–111. Oxford University Press, 1998.

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of Univalent Foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23(6):2071–2126, March 2021.

[3] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[4] Daniel R. Licata and Robert Harper. 2-Dimensional Directed Type Theory. *Electronic Notes in Theoretical Computer Science*, 276:263–289, September 2011.

[5] Nuyts Andreas. Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance. Master's thesis, KU Leuven, 2015.

[6] Daniel Gratzer, Jonathan Weinberger, and Ulrik Buchholtz. Directed univalence in simplicial homotopy type theory, 2024. Version Number: 1.

[7] Thorsten Altenkirch and Jacob Neumann. Synthetic 1-Categories in Directed Type Theory, 2024. Version Number: 1.

[8] Emily Riehl and Michael Shulman. A type theory for synthetic ∞-categories. *Higher Structures*, 1(1):147–224, December 2017.

[9] Chu, Fernando, Mangel, Éléonore, and North, Paige Randall. A directed type theory for 1-categories. In *30th International Conference on Types for Proofs and Programs (TYPES 2024)*, 2024.

[10] Paige Randall North. Towards a Directed Homotopy Type Theory. *Electronic Notes in Theoretical Computer Science*, 347:223–239, November 2019.

[11] Ross Street. Fibrations in bicategories. *Cahiers de topologie et géométrie différentielle*, 21(2):111–160, 1980. (Corrections in 28(1):53–56, 1987).