Directed equality with dinaturality

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Equality in Martin-Löf type theory is inherently symmetric [8]: this is what allows for types to be interpreted as sets, groupoids [9], and ∞-groupoids [19]; points of a type correspond to objects, and equality is precisely interpreted by morphisms which are always invertible.

A natural question follows: can there be a variant of Martin-Löf type theory which enables types to be interpreted as *categories*, where morphisms need not be invertible? Such a system should take the name of *directed type theory* [13, 16, 1, 6, 11, 2] (DTT), where the directed aspect comes from a non-symmetric interpretation of "equality", which now has a source and a target in the same way that morphisms do in a category. A common feature of current semantic approaches to directed type theory is to resort back to the maximal sub*groupoid* $\mathbb{C}^{\mathsf{core}}$ of categories [16, 2] in order to use the same variable with different variances $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$; this is needed to validate introduction (refl) and elimination (J) rules for directed equality.

Dinaturality for directed type theory. In this work, we describe a first-order non-dependent proof-relevant type theory where types are interpreted as categories, terms as functors, predicates as endoprofunctors and entailments as *dinatural transformations* [4]; intuitively, dinatural transformations allow for the same variable to appear both covariantly and contravariantly, and terms are required to be given only "on the diagonal" by equating the two occurrences with the same value, thus avoiding the need for groupoids. An excerpt of the rules of our type theory is shown in Figure 1, where (refl) and (J) capture the rules for directed equality. This type theory is equipped with a syntactic notion of polarity which allows variables to be distinguished based on their appearance in negative and positive positions, which are represented in entailments as $\overline{x} : \mathbb{C}^{\mathsf{op}}$ and $x : \mathbb{C}$, respectively. Such polarity is used to express a syntactic requirement on the J-rule: given a directed equality in context $\hom_{\mathbb{C}}(x, y)$ with $x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C}$, then x and y are allowed to be contracted to the same variable z only if both x and y appear only positively (i.e., with the same polarity) in the conclusion and only negatively (i.e., with the opposite polarity) in the context. This rule allows us to syntactically recover the same definitions about equality that one expects in standard Martin-Löf type theory, *except for symmetry* because of the syntactic restrictions: e.g., transitivity of directed equality (the composition map in a category), congruences of terms along directed equalities (the action of a functor on morphisms), transport along directed equalities (the coYoneda lemma), and an internal (di)naturality statement. Proving equational properties about these maps also follows the same steps as in Martin-Löf type theory using *directed equality induction*, given in (J-eq), which is a "dependent" version of (J) for the judgement of equalities of entailments. Crucially, (J-eq) is validated in the model using dinaturality of maps. As in the case of symmetric equality [10, Lemma 3.2.3], (J) is actually an isomorphism, and the inverse map is given by precomposing with (refl). Finally, the interval type $\mathbf{I} := \{0 \to 1\}$ with a single arrow serves as countermodel for symmetry of directed equality.

Dinaturals famously do not compose [18]; the practical consequence of this fact is that in this type theory there is no general cut rule for entailments. We provide two restricted cut rules (cut-din) and (cut-nat), intuitively capturing the composition of dinaturals with *naturals*, which are enough to capture all practical cases in which composition is needed. The rule (cut-assoc) captures associativity of these two compositions in the equational theory for entailments.

*Loregian was supported by the Estonian Research Council grant PRG1210. Veltri was supported by the Estonian Research Council grant PSG749.

$$\boxed{[\Gamma]\ \Phi \vdash \alpha : P} \qquad \frac{}{[\Gamma]\ \Phi, a : P, \Phi' \vdash a : P}\ (\mathsf{var}) \qquad \frac{[\Gamma]\ \Phi \vdash \alpha : P}{[\Gamma]\ A, \Phi \vdash \mathsf{wk}(\alpha) : P}\ (\mathsf{wk}) \qquad \frac{}{[\Gamma]\ \Phi \vdash ! : \top}\ (\top)$$

$$\frac{[x : \mathbb{C}, \Gamma]\ \Phi(\overline{x}, x) \vdash \alpha : P(\overline{x}, x)}{[x : \mathbb{C}^{\mathsf{op}}, \Gamma]\ \Phi(x, \overline{x}) \vdash \alpha : P(x, \overline{x})}\ (\mathsf{op}) \qquad \frac{\Gamma \vdash F : \mathbb{C} \qquad [x : \mathbb{C}, \Gamma]\ \Phi(\overline{x}, x) \vdash \alpha : Q(\overline{x}, x)}{[\Gamma]\ \Phi(F^{\mathsf{op}}(\overline{x}), F(x)) \vdash F^*(\alpha) : Q(F^{\mathsf{op}}(\overline{x}), F(x))}\ (\mathsf{idx})$$

$$\frac{[\Gamma]\ \Phi \vdash P \times Q}{[\Gamma]\ \Phi \vdash P, \qquad [\Gamma]\ \Phi \vdash Q}\ (\mathsf{prod}) \qquad \frac{[x : \Gamma]\ A(\overline{x}, x), \Phi(\overline{x}, x) \vdash B(\overline{x}, x)}{[x : \Gamma]\qquad \Phi(\overline{x}, x) \vdash A^{\mathsf{op}}(x, \overline{x}) \Rightarrow B(\overline{x}, x)}\ (\mathsf{exp})$$

$$\frac{\begin{array}{r}[a : \Delta^{\mathsf{op}}, b : \Delta, x : \Gamma]\ \Phi(\overline{x}, x, a, b) \vdash \alpha \quad : P(a, b)\\ [z : \Delta, x : \Gamma]\ k : P(\overline{z}, z), \Phi(\overline{x}, x, \overline{z}, z) \vdash \gamma[k] : Q(\overline{z}, z)\end{array}}{[z : \Delta, x : \Gamma]\ \Phi(\overline{x}, x, \overline{z}, z) \vdash \gamma[\alpha] : Q(\overline{z}, z)}\ (\mathsf{cut\text{-}din})$$

$$\frac{\begin{array}{r}[z : \Delta, x : \Gamma]\ \Phi(\overline{x}, x, \overline{z}, z) \vdash \gamma \quad : P(\overline{z}, z)\\ [a : \Delta^{\mathsf{op}}, b : \Delta, x : \Gamma]\ k : P(a, b), \Phi(\overline{x}, x, \overline{a}, \overline{b}) \vdash \alpha[k] : Q(a, b)\end{array}}{[z : \Delta, x : \Gamma]\ \Phi(\overline{x}, x, \overline{z}, z) \vdash \alpha[\gamma] : Q(\overline{z}, z)}\ (\mathsf{cut\text{-}nat})$$

$$\frac{}{[x : \mathbb{C}, \Gamma]\ \Phi \vdash \mathsf{refl}_{\mathbb{C}} : \hom_{\mathbb{C}}(\overline{x}, x)}\ (\mathsf{refl}) \qquad \frac{[z : \mathbb{C}, \Gamma]\qquad \Phi(\overline{z}, z) \vdash h : P(\overline{z}, z)}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, \Gamma]\ e : \hom_{\mathbb{C}}(a, b), \Phi(\overline{b}, \overline{a}) \vdash J(h)[e] : P(a, b)}\ (J)$$

$$\frac{[a : \mathbb{C}, \Gamma]\ \Phi \vdash Q(\overline{a}, a)}{[\Gamma]\ \Phi \vdash \int_{a:\mathbb{C}} Q(\overline{a}, a)}\ (\mathsf{end}) \qquad \frac{[\Gamma]\ \left(\int^{a:\mathbb{C}} Q(\overline{a}, a)\right), \Phi \vdash P}{[a : \mathbb{C}, \Gamma]\ Q(\overline{a}, a), \Phi \vdash P}\ (\mathsf{coend})$$

$$\boxed{[\Gamma]\ \Phi \vdash \alpha = \beta : P} \qquad \frac{}{[z : \mathbb{C}, \Gamma]\ k : \Phi(\overline{z}, z) \vdash J(h)[\mathsf{refl}_{\mathbb{C}}] = h : P(\overline{z}, z)}\ (J\text{-}\mathsf{comp})$$

$$\frac{[z : \mathbb{C}, \Gamma]\ \Phi(\overline{z}, z) \vdash \alpha[\mathsf{refl}_{\mathbb{C}}] = \beta[\mathsf{refl}_{\mathbb{C}}] : P(\overline{z}, z)}{[a : \mathbb{C}^{\mathsf{op}}, b : \mathbb{C}, \Gamma]\ e : \hom_{\mathbb{C}}(a, b), \Phi(\overline{b}, \overline{a}) \vdash \alpha[e] = \beta[e] : P(a, b)}\ (J\text{-}\mathsf{eq})$$

$$\frac{\begin{array}{r}[a : \Delta^{\mathsf{op}}, b : \Delta, x : \Gamma]\qquad \Phi(\overline{x}, x, a, b) \vdash \alpha \quad : P(a, b)\\ [z : \Delta, x : \Gamma]\ k : P(\overline{z}, z), \Phi(\overline{x}, x, \overline{z}, z) \vdash \gamma[k]\ : Q(\overline{z}, z)\\ [a : \Delta^{\mathsf{op}}, b : \Delta, x : \Gamma]\ k : Q(a, b), \Phi(\overline{x}, x, \overline{a}, \overline{b}) \vdash \beta[k] : R(a, b)\end{array}}{[z : \Delta, x : \Gamma]\qquad \Phi(\overline{x}, x, \overline{z}, z) \vdash (\beta[\gamma])[\alpha] = \beta[\gamma[\alpha]] : R(\overline{z}, z)}\ (\mathsf{cut\text{-}assoc})$$

Figure 1: Main rules for entailments of first-order dinatural directed type theory.

Implication in the logic is given by the notion of *polarized exponentials* [5, 3], which are interpreted via the pointwise hom of endoprofunctors in Set: their behaviour is captured in the rule (exp), which intuitively allows for predicates to switch between the two sides of the turnstile simply by inverting the variance of all their variables.

(Co)ends as quantifiers. Moreover, we show how dinaturality allows us to more precisely view (co)ends [14] as the "directed quantifiers" of directed type theory, which we present in a correspondence reminiscent of the quantifiers-as-adjoints paradigm of Lawvere [12]. The rules for (co)ends are captured as (coend), (end). Despite the lack of general composition, the rules for directed equality and coends-as-quantifiers can be used to give concise proofs of theorems in category theory using a distinctly *logical* flavour via a series of isomorphisms: e.g., the (co)Yoneda lemma, Kan extensions computed via (co)ends are adjoints, presheaves form a closed category, hom preserves (co)limits, and Fubini, which easily follow by modularly using the logical rules of each connective. This constitutes a concrete step towards formally understanding the so-called "(co)end calculus" [14] from a logical perspective. We show an example of (co)end calculus as directed type theory via the following two derivations, which formally capture the Yoneda and coYoneda lemma, respectively, using the rules given in Figure 1:

$$\dfrac{\dfrac{\dfrac{[a:\mathbb{C}]\ \Phi(a) \vdash \int_{x:\mathbb{C}} \hom^{\mathsf{op}}_{\mathbb{C}}(a,\overline{x}) \Rightarrow P(x)}{[a:\mathbb{C}, x:\mathbb{C}]\ \Phi(a) \vdash \hom^{\mathsf{op}}_{\mathbb{C}}(a,\overline{x}) \Rightarrow P(x)}\ (\mathsf{end})}{[a:\mathbb{C}]\ \hom_{\mathbb{C}}(\overline{a},x) \times \Phi(a) \vdash P(x)}\ (\mathsf{exp})}{[z:\mathbb{C}]\ \Phi(z) \vdash P(z)}\ (J)
\qquad
\dfrac{\dfrac{[a:\mathbb{C}]\ \int^{x:\mathbb{C}} \hom_{\mathbb{C}}(\overline{x},a) \times P(x) \vdash \Phi(a)}{[a:\mathbb{C}, x:\mathbb{C}]\ \hom_{\mathbb{C}}(\overline{a},x) \times P(a) \vdash \Phi(x)}\ (\mathsf{coend})}{[z:\mathbb{C}]\ P(z) \vdash \Phi(z)}\ (J)$$

Related works. North [16] describes a dependent directed type theory with semantics in Cat, but using groupoidal structure to deal with the problem of variance in both introduction and elimination rules for directed equality. A similar approach is followed in [2] using the notion of neutral contexts (i.e. groupoidal) instead of core-types. We focus on non-dependent semantics, and tackle the variance issue precisely with the notion of dinatural transformation instead of having to resort back to groupoidal structure.

New and Licata [15] give a sound and complete presentation for the internal language of (hyperdoctrines of) certain virtual equipments. These models capture enriched, internal, and fibered categories, and have an intrinsically directed flavour. This generality comes at the cost of a non-standard syntactic structure of the logic, which forces variables to appear in an ordered linear way. Our work is similar in spirit since we provide a formal setting to prove category-theoretical theorems using logical methods, but we only focus on the 1-category model. We treat ends and coends as quantifiers *directly*, with adjoint-like correspondences to weakening functors acting on the context, without the need for quantifiers to include (restricted forms of) conjunction and implication as in their work. Our rules for directed equality are more reminiscent of the J-rule in MLTT, and specifically focus on the semantic justification based on dinaturality; because of dinaturality, we are similarly restricted in the way that composition can be performed, and indeed the specific substitution structure of Prof, viewed as a double category, is an instance of our cut rules. Since we consider less general models, our contexts do not have any restriction on appearance of variables: this allows us to consider profunctors of many variables as typically needed in (co)end calculus (e.g. Fubini) and to *state* the statement that directed equality can be symmetric. Moreover, certain derivations, e.g., the fact that presheaf categories are cartesian closed, are not easy to capture as abstract properties of such models, while they are straightforward to capture logically using dinaturality.

Another approach to directed type theory involves using geometric models with different flavours [17, 6, 7, 20], typically by axiomatizing a directed interval type. In comparison, we work directly with the more "algebraic" and elementary notions of 1-categories and dinatural transformations, interpreting directed equality directly with hom-functors and their elimination rules rather than with synthetic intervals.

Directed equality can also be captured at the judgemental level [13, 1]; however, such rewrites are typically not described internally using hom-types and their elimination rule.

Future work. We believe type dependency to be the most interesting development of this theory, with particular attention to how the polarity of variables interacts with type dependency. To that end we are currently exploring a notion of *dinatural context extension* based on a "diagonal" generalization of the Grothendieck construction to dinatural families of types $A : \Gamma^{\mathsf{op}} \times \Gamma \to \mathsf{Cat}$, where objects are pairs $(X \in \Gamma, a \in A(X,X))$, and arrows are given by a lax wedge-like condition as pairs $(X,a) \to (Y,b) := (f : X \to Y, \alpha : A(\mathsf{id}_X, f)(a) \to A(f, \mathsf{id}_Y)(b))$.

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