Weihrauch problems as containers

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Abstract

We note that Weihrauch problems can be regarded as containers over the category of projective represented spaces and that Weihrauch reductions correspond exactly to container morphisms. We also show that Bauer's extended Weihrauch degrees and the posetal reflection of containers over partition assemblies are equivalent. Using this characterization, we show how a number of operators over Weihrauch degrees, such as the composition product, also arise naturally from the abstract theory of polynomial functors.

The content discussed in this abstract is from [13].

Weihrauch Reducibility Weihrauch reducibility is a framework from computable analysis for comparing the computational strength of partial multi-valued functions over Baire space, i.e., relations on $\mathbb{N}^{\mathbb{N}}$, which are thus called *Weihrauch problems*. Intuitively, a problem P is reducible to a problem Q, written $P \leq_{\mathrm{W}} G$, if we can compute P given an oracle for Q that can be called once.

Definition 1 ([10]). *If P and G are two Weihrauch problems, P is said to be* Weihrauch reducible to G *if there exist partial type 2 computable*[1] *maps φ and ψ such that φ is a map $\mathrm{dom}(P) \to \mathrm{dom}(G)$ and for every $i \in \mathrm{dom}(P)$ and $j \in G(\varphi(i))$, $\psi(i, j)$ is defined and belongs to $P(i)$.*

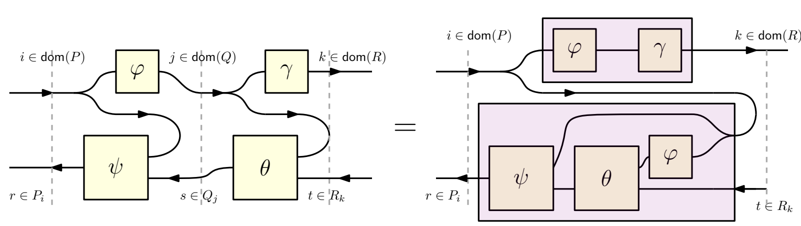
Example 1. *We can encode the following as Weihrauch problems:*

- LPO *("given a bit sequence, tell me if it has a 1"), defined by*

$$\mathsf{LPO}(p) = \{0^\omega \mid p = 0^\omega\} \cup \{1^\omega \mid p \in \{0,1\}^{\mathbb{N}}, p \neq 0^\omega\}$$

- KL*: "given an infinite finitely branching tree t, give me an infinite path through t" can also be encoded as a Weihrauch problem, modulo a standard embedding $2^{\mathbb{N}^{<\omega}} \hookrightarrow \mathbb{N}^{\mathbb{N}}$.*

LPO *is Weihrauch reducible to* KL*, but not* KL *is not reducible to* LPO*.*



Since Weihrauch reductions compose as depicted above, Weihrauch problems and reductions form a preorder. The equivalence classes (called Weihrauch degrees) form a distributive lattice, where meets and joins can be regarded as natural operators on problems: the join $P \sqcup Q$ allows to ask a question either to P or Q and get the relevant answer, while $P \sqcap Q$ requires to ask one

[1] Type-2 computable maps are partial computable stream transformers $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

question to each and get only one answer. Many other natural operators have been introduced on the Weihrauch lattice, including a parallel product $P \times Q$ (ask questions to both P and Q, get both answers), a composition operator $P \star Q$ (ask a question to Q and then, depending on the answer you got, a question to P) and many others, including residuals and fixpoints of other operators [5, Definition 1.2].

Containers Intuitively, a container in a category $\mathcal{C}$ is a family of objects of $\mathcal{C}$ indexed by an object of $\mathcal{C}$. Since objects of $\mathcal{C}$ are not necessarily sets, to make this formal, we define *containers* as objects of ${}^{\mathcal{C}}/_{I}$. Morphisms of containers are defined as follows.

Definition 2. *A morphism representative from a container $P : X \to U$ to $Q : Y \to V$ is a pair (φ, ψ) making the following diagram commute, the rightmost square being a pullback:*

$$\begin{array}{ccccc} X & \xleftarrow{\ \psi\ } & \sum_{u \in U} Y_{\varphi(u)} & \longrightarrow & Y \\ {\scriptstyle P}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle Q} \\ U & = & U & \xrightarrow[\ \varphi\ ]{} & V \end{array}$$

A morphism of containers is an equivalence class of morphism representatives.

The category of containers crops up in several places in the literature on functional programming and mathematics and are sometimes referred to as polynomials or dependent lenses [1, 6, 17]. It is a "category of abundance", having four monoidal structures (products, coproducts, $\otimes$, and a composition $\circ$) and many other desirable properties [11, 14].

Contributions The representation of container morphisms into a "forward map" ϕ and a family of "backward maps" ψ, as well as similar types of algebraic structures in the poset of Weihrauch degrees and the category of containers suggests a strong relation between the two. Our main contribution is to make that connection formal: Weihrauch degrees are isomorphic to the posetal reflection of the full subcategory of *answerable* containers over the category $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$ of partitioned represented spaces and (type 2) computable functions. This is technically straightforward: one regards a container $P : X \to U$ as the problem of finding a section to P and morphisms as reductions between problems. P being answerable intuitively means that a (possibly non-computable!) section of P exists at all[2].

Thus a similar correspondence also applies to close variants such as continuous Weihrauch reducibility and extended Weihrauch degrees [3]. Our most recent ongoing work is characterising strong Weihrauch reducibility in a similar way via *dependent adaptors*[3].

Coproducts and products of containers correspond to the lattice operators in the Weihrauch degrees. The tensor product $\otimes$ of containers corresponds to the parallel product of Weihrauch degrees. The treatment of composition of containers and composition of Weihrauch degrees is trickier. The reason for this is that Weihrauch problem correspond to containers over a category which is only *weakly* (locally) cartesian closed. This means that we while we may define a composition operator, it will only be a quasi-bifunctor.

A summary of the current state of what we know and conjecture is given in table 1, for details see our preprint [13].

[2] In general, starting from any category $\mathcal{C}$ with pullbacks, we can say that P is answerable iff it is a pullback-stable epimorphism and derive the expected basic properties of $\times, +$ and $\otimes$ [13, §4.1].

[3] We adopt the definition and terminology from a seminar talk given by Hedges [4] – we are currently not aware of another citable source where the notion might have been spelled out.

Containers	Reduciblity	Status
Answerable containers over $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$	Weihrauch problems	✓
Containers over $\mathsf{pAsm}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$	Extended Weihrauch problems	✓
Dependent adaptors over $\mathsf{pMod}(\mathcal{K}_2^{\mathrm{rec}}, \mathcal{K}_2)$	Strong Weihrauch problems	✓
Product $p \times q$	Meet $p \sqcap q$	✓
Coproduct $p + q$	Join $p \sqcup q$	✓
Tensor product $p \otimes q$	Parallel product $p \times q$	✓
Composition product $p \triangleleft q$	Composition product $p \star q$	✓*
Free monad on p	Iterated composition product $p^\diamond$	
Closed structures $\multimap, \Rightarrow$	?	
Derivative ∂p	?	
?	First-order part ${}^1p$	
?	Deterministic part $\det(p)$	
...	...	

Table 1: Relating container concepts to reducibility concepts.

Related work The idea of regarding a bundle as a problem to be solved by finding one of its section is an old one that predates the "container" terminology. For instance, Hirsch [7, Definition 3.4] defines an equivalent category to study the topological complexity of problems and reductions between those. This perspective also already appeared in the literature on Weihrauch problems (see for instance [9, Remark 2.8]), although most of the recent efforts we are aware of to "categorify" Weihrauch reducibility and operators on problems tend to use other tools instead.

One natural categorical construction that captured Weihrauch degrees that appeared in the literature is the restriction of Bauer's extended Weihrauch problems [3] to objects that are actually Weihrauch problems, which are characterized as the $\neg\neg$-dense predicates over modest sets. Interestingly, Ahman and Bauer also linked extended Weihrauch reducibility to containers [2], but by way of the more general notion of instance reducibility that works over families of truth values. while here we work directly with bundles of partitioned assemblies.

Trotta et. al. also formally linked (extended) Weihrauch reducibility with the Dialectica interpretation [15], which can be regarded bicompletions of fibrations by simple products and sums [8]. Aside from the fact that they work in a posetal setting throughout, it is interesting to note that the category of containers over $\mathcal{C}$ can be recovered by completing the terminal fibration over $\mathcal{C}$ by arbitrary products and then sums and taking the fiber over 1. The Dialectica interpretation was also used by Uftring [16] to capture Weihrauch reducibility in a syntactic way in a substructural arithmetic.

Pauly also studied a generic notion of reducibility that encompasses Weihrauch reducibility, starting from categories of multivalued functions [12], in which he derived the lattice operators as well as finite parallelizations in a generic way.

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