Cohomology in Synthetic Stone Duality

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Peter Scholze and Dustin Clausen [CS24] introduced light condensed sets, defined as sheaves on the site of light profinite sets. They can be used as an alternative to topological spaces. Synthetic Stone duality is an extension of homotopy type theory by four axioms, which was introduced in [Che+24]. In this article, it was proven that $H^1(S, \mathbb{Z}) = 0$ for S a Stone space, that $H^1(X, \mathbb{Z})$ for X compact Hausdorff can be computed using Čech cohomology and that $H^1(\mathbb{I}, \mathbb{Z}) = 0$ where $\mathbb{I}$ is the unit interval. In this talk we will present the extension of these results to higher cohomology groups with non-constant countably presented abelian groups as coefficients. Those are synthetic analogues of results from Roy Dyckhoff [Dyc76a; Dyc76b].

Synthetic Stone Duality In our setting, Stone spaces are precisely *countable* sequential limits of finite sets, making them analogous to *light* profinite sets. The construction of a model making this analogy rigorous is still work in progress.

The axioms of synthetic Stone duality postulate Stone duality (Stone spaces are equivalent to countably presented Boolean algebras), completeness (non-empty Stone spaces are merely inhabited), dependent choice and a *local-choice* axiom. The latter says that given a Stone space S and a type family B over S such that $\Pi_{x:S}\|B(x)\|$, there merely is a Stone space T and a surjection $s : T \twoheadrightarrow S$ such that $\Pi_{x:T}B(s(x))$. Local choice is crucial when performing the cohomological computations mentioned below.

Many traditional properties of Stone spaces can be shown synthetically, sometimes phrased in a more type-theoretic way, e.g. Stone spaces are closed under Σ-types. Open and closed propositions can be defined, inducing a topology on any type such that any map is continuous. This topology is as expected for Stone spaces and compact Hausdorff spaces (i.e. quotient of Stone spaces by closed equivalence relations).

One important example of compact Hausdorff space is the real interval $\mathbb{I}$, from which the type $\mathbb{R}$ of real numbers is constructed. This is equivalent to the usual constructions of both Cauchy and Dedekind reals. As in [Shu18], it is important to distinguish topological spaces like $\mathbb{S}^1 := \{x, y : \mathbb{R} \mid x^2 + y^2 = 1\}$ from homotopical spaces like the higher inductive 1-type S^1.

Despite topological spaces being sets, they can have non-trivial cohomology. Indeed, for any type X and dependent abelian group $A : X \to \mathrm{Ab}$, we use the usual synthetic definition of the n-th cohomology group $H^n(X, A)$ as $\|\Pi_{x:X}K(A_x, n)\|_0$ where $K(A_x, n)$ is the n-th Eilenberg Mac-Lane space with coefficient A_x. In [Che+24], it is proven that $H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$, despite $\mathbb{S}^1$ being a set.

We prove Barton and Commelin's condensed type theory axioms [Bar24] in synthetic Stone duality, as well as dependent generalisations of them. These are used to show that any compact Hausdorff space X interact well with any family of countably presented abelian groups $A : X \to \mathrm{Ab}_{\mathrm{cp}}$.

Vanishing of higher cohomology of Stone spaces First we prove that $H^1(S, A) = 0$ for all S Stone and $A : S \to \mathrm{Ab}_{\mathrm{cp}}$. We assume $\alpha : \prod_{x:S} K(A_x, 1)$, by local choice we get a surjection $p : T \twoheadrightarrow S$ with T Stone which trivialises α. Then we get an approximation of p as a sequential limit of surjective maps $p_k : T_k \to S_k$ between finite sets, we check that the induced trivialisation

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over T_k gives a trivialisation over S_k, and conclude through our dependent generalisations of Barton and Commelin's axioms that α is trivial over $\lim_k S_k = S$.

We follow an idea due to David Wärn [BCW23, Theorem 3.4] to go from $H^1(S, A) = 0$ to $H^n(S, A) = 0$ for all $n > 0$. The key idea is to proceed by induction on $n > 0$, generalising the induction hypothesis from $H^n(S, A) = 0$ to:

(i) $K(\prod_{x:S} A_x, n) \to \prod_{x:S} K(A_x, n)$ is an equivalence, directly implying $H^n(S, A) = 0$.

(ii) $K(\prod_{x:S} A_x, n+1) \to \prod_{x:S} K(A_x, n+1)$ is an embedding.

Assume (i) and (ii) for $n > 0$, let's prove (i) and (ii) for $n+1$. (ii) follows immediately from (i). To prove (i), by induction hypothesis (ii) it is enough to prove that $\prod_{x:S} K(A_x, n+1)$ is connected, i.e. $H^{n+1}(S, A) = 0$. We assume $\alpha : \prod_{x:S} K(A_x, n+1)$, by local choice we get a trivialisation $p : T \twoheadrightarrow S$ of α with T Stone. Denoting by T_x the fiber of p over x, we consider the exact sequence $0 \to A_x \to A_x^{T_x} \to L_x \to 0$ giving an exact sequence:

$$H^n(S, L) \to H^{n+1}(S, A) \to H^{n+1}(S, \lambda x.A_x^{T_x})$$

By induction hypothesis (i) we have that $H^n(S, L) = 0$ so we have an injection:

$$H^{n+1}(S, A) \to H^{n+1}(S, \lambda x.A_x^{T_x})$$

By induction hypothesis (ii) the map:

$$H^{n+1}(S, \lambda x.A_x^{T_x}) \to H^{n+1}(\Sigma_{x:S} T_x, A_x) = H^{n+1}(T, A \circ p)$$

is an injection so that we get an injection:

$$p^* : H^{n+1}(S, A) \to H^{n+1}(T, A \circ p)$$

But p trivialises α so $p^*(\alpha) = 0$, therefore $\alpha = 0$.

Čech cohomology for compact Hausdorff spaces Given a compact Hausdorff space X, a Čech cover for X consists of a Stone space S and a surjective map $p : S \twoheadrightarrow X$. By definition any compact Hausdorff space has a Čech cover.

Given such a Čech cover and $A : X \to \mathrm{Ab}_{\mathrm{cp}}$, we define its Čech complex as:

$$\Pi_{x:X} A_x^{S_x} \to \Pi_{x:X} A_x^{S_x \times S_x} \to \cdots$$

with the boundary maps defined as expected, and its Čech cohomology $\check{H}^k(X, A)$ as the k-th homology group of its Čech complex. It is clear that $H^0(X, A) = \check{H}^0(X, A)$.

From hypothesis (i) in the previous paragraph, for $n > 0$ we get an exact sequences:

$$H^{n-1}(X, \lambda x.A_x^{S_x}) \to H^{n-1}(X, L) \to H^n(X, A) \to 0$$

By direct computations, for $n > 0$ we get an exact sequence:

$$\check{H}^{n-1}(X, \lambda x.A_x^{S_x}) \to \check{H}^{n-1}(X, L) \to \check{H}^n(X, A) \to 0$$

We conclude by induction on n that $H^n(X, A) = \check{H}^n(X, A)$ for all n, so that in particular Čech cohomology does not depend on the Čech cover. For this induction to go through it is crucial that $A_x^{S_x}$ is countably presented, which follows from Barton and Commelin's axioms. Using this result and finite approximations of a well-chosen Čech cover of $\mathbb{I}$, we can check that $H^n(\mathbb{I}, A) = 0$ for all $n > 0$ and $A : \mathbb{I} \to \mathrm{Ab}_{\mathrm{cp}}$.

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