Constructive algebraic completeness of first-order bi-intuitionistic logic

Dominik Kirst[1] and Ian Shillito[2]

[1] Université Paris Cité, IRIF, Inria, Paris, France
dominik.kirst@inria.fr
[2] University of Birmingham, Birmingham, United Kingdom
i.b.p.shillito@bham.ac.uk

Background Bi-intuitionistic logic extends intuitionistic logic with the *exclusion* binary operator $\prec$, dual to implication. This extension is mathematically natural, as symmetry is regained in the language: each operator has a dual, even $\neg\varphi := \varphi \to \bot$ has $\sim\varphi := \top \prec \varphi$. However, bi-intuitionistic logic is packed with surprises. First, it proves a bi-intuitionistic version of LEM ($\varphi \vee \sim\varphi$), and is thus not constructive as it fails the disjunction property. Second, despite not being constructive it does not yet collapse to classical logic, given that it is a conservative extension of intuitionistic logic (in the propositional case). Third, this logic bears a striking resemblance to modal logic, as it splits into a *local* and *global* logic[1] and is tightly connected to the tense logic S4t [19, 20, 21]. Finally, the conservativity of bi-intuitionistic logic over intuitionistic logic only holds in the propositional case: the *constant domain axiom* $(\forall x(\varphi(x) \vee \psi) \to (\forall x \varphi(x) \vee \psi))$ is provable in *first-order* bi-intuitionistic logic, but not intuitionistically.

Historically, bi-intuitionistic logic was developed by Cecylia Rauszer in a series of articles in the 1970s [25, 24, 26, 27] leading to her Ph.D. thesis [28].[2] She studied this logic under a wide variety of aspects: algebraic semantics, axiomatic calculus, sequent calculus, and Kripke semantics. Unfortunately, through time a variety of mistakes have been detected in her work. First, Crolard proved in 2001 that bi-intuitionistic logic is not complete for the class of rooted frames [2, Corollary 2.18], in contradiction with Rauszer's proof which relies on rooted canonical models. Second, Pinto and Uustalu found in 2009 [18] a counterexample to Rauszer's claim of admissibility of cut for the sequent calculus she designed [24, Theorem 2.4]. Finally, Goré and Shillito in 2020 [7] detected confusions in Rauszer's work about the holding of the deduction theorem in bi-intuitionistic logic, as well as issues in her completeness proofs.

The foundations of bi-intuitionistic logic have therefore been in reconstruction since these discoveries. In the propositional case serious progress has been made: sequent calculi for the local logic were provided [18]; axiomatic calculi and Kripke semantics were connected [7, 29] and reverse analysed in a constructive setting [30]; the algebraic treatment was recently completed by Deakin and Shillito [4, 3]. As for the first-order case, it has received little attention until recently, where it was shown to fail Craig interpolation [17], studied via polytree labelled sequent calculi [13], and its soundness and completeness w.r.t. its Kripke semantics were established [10].

No reconstruction of the algebraic treatment of first-order bi-intuitionistic logic has yet been provided. We fill this gap by proving that the local first-order bi-intuitionistic logic is *sound* and *weakly complete* with respect to bi-Heyting algebras: $\vdash \varphi$ iff $\vDash \varphi$. Our work is motivated by foundational interests, but also by constructive sensitivity: these algebraic results usually hold constructively (cf. [5]), while our previous proofs [10] for the Kripke semantics use classical principles and certainly necessarily so, given the already established propositional analysis [30].

[1] This terminology comes from the Kripke semantics, in which one can define a local and a global notion of semantic consequence [1, Section 1.5].

[2] While appearances of bi-intuitionistic can be detected prior to these articles, notably in 1942 in Moisil's work [15], in 1964 in Grzegorczyk's work [8], and in 1971 in Klemke's work [11], the breadth and foundational nature of Rauszer's work advocate for her place as founder of this logic.

Algebraic interpretation of quantifiers The algebraic interpretation of the propositional bi-intuitionistic connectives straightforwardly extends the one for intuitionistic logic: Heyting algebras, i.e. bounded distributive lattices with an implication satisfying the residuation on the left below, are extended to bi-Heyting algebras with the exclusion operator satisfying the dual residuation on the right.

$$\frac{\varphi \wedge \psi \leq \chi}{\varphi \leq \psi \to \chi} \qquad\qquad \frac{\varphi \leq \psi \vee \chi}{\varphi \prec \psi \leq \chi}$$

However, the interpretation of quantifiers in algebraic semantics is famously not obvious. Some approaches involve complex algebraically inspired structures, like hyperdoctrines [12] or cylindric algebras [16]. Instead, we follow a tradition [23, 22] of grafting interpretation of quantifiers on the base algebras (bi-Heyting in our case) via the creation of *infima* $\bigsqcap$, interpreting $\forall$, and *suprema* $\bigsqcup$, interpreting $\exists$. The strikingly obvious issue with this approach is that some bi-Heyting algebras do not have such infinite elements. We simply discard these algebras and focus on the class of *complete* bi-Heyting algebras, i.e. those possessing these infinite elements.

MacNeille completion of the Lindenbaum-Tarski algebra Traditionally, one proves (weak) completeness by building a so-called Lindenbaum-Tarski (LT) algebra for the logic, a syntactic algebra made out of equivalence classes of formulas $[\![\varphi]\!] := \{\psi \mid \vdash \varphi \leftrightarrow \psi\}$ in which $\leq$ captures entailment: $[\![\varphi]\!] \leq [\![\psi]\!]$ iff $\varphi \vdash \psi$. This algebra is easily defined for propositional bi-intuitionistic logic, constituting the ideal candidate to graft quantifiers onto to prove completeness. Sadly, we are not ensured that this algebra is complete, hence it may then be outside of the class of complete algebras under focus. To overcome this difficulty, we complete the LT algebra via the *MacNeille completion* [14], which embeds a partially ordered set $(X, \leq)$ into the complete lattice of all its subsets A which are *successively* closed under upper bounds and lower bounds $((A^u)^l \subseteq A)$. In the resulting algebra, $\to$ receives a natural interpretation giving rise to a Heyting algebra [22, 9], leaving us with the task of interpreting $\prec$ to obtain completeness. As indicated by Harding and Bezhanishvili [9], this interpretation is not natural:

$$X \prec Y := \{x \mid \forall y \in Y^u.\ (y \vee x) \in X^u\}^l$$

Still, we obtain a complete bi-Heyting algebra embedding the initial propositional LT algebra, establishing weak completeness.

A Pathway to strong completeness Ideally, we would like to prove *strong* completeness, i.e. $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$, with respect to the following algebraic consequence relation, where φ^* is the interpretation of φ in A via the valuation V mapping atomic formulas to elements of X.

$$\Gamma \vDash \varphi \qquad := \qquad \forall A.\ \forall V.\ \forall a \in A.\ \ (\forall \gamma \in \Gamma.\ a \leq \gamma^*) \quad \to \quad a \leq \varphi^*$$

To prove strong completeness, we want to generate a proof of $\Gamma \vdash \varphi$ from $\Gamma \vDash \varphi$ using the completed LT algebra. It would then suffice to find an element below the interpretation of all $\gamma \in \Gamma$. An obvious candidate is the (existing) infimum $\bigsqcap \Gamma^*$, which is below φ^* under the assumption $\Gamma \vDash \varphi$. Unfortunately, $\bigsqcap \Gamma$ is not a formula in our language, so we cannot extract $\bigsqcap \Gamma \vdash \varphi$ from $\bigsqcap \Gamma^* \leq \varphi^*$. If the infimum of Γ^* was *compact*, i.e. such that $\bigsqcap \Gamma^* \leq a$ entails the existence of a *finite* subset $\Delta^* \subseteq \Gamma^*$ with $\bigsqcap \Delta^* \leq a$, we would be able to close the strong completeness proof. A way to obtain compact infima and suprema is through *canonical extensions* [6]. In further work we will inspect the interplay between MacNeille completion for bi-Heyting algebras and canonical extension, in the hope to deduce strong completeness.

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