Yet another homotopy group, yet another Brunerie number

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Homotopy type theory (HoTT) has been proposed as a foundation of synthetic homotopy theory and its usefulness was witnessed early on by e.g. Brunerie's 2016 proof that $\pi_4(\mathbb{S}^3)$ – the fourth homotopy group of the 3-sphere – is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [Bru16]. As Brunerie's proof appeared only 78 years after Pontrjagin's original proof [Pon38], it is about time (if we wish to keep up with the classical homotopy theorists) we start thinking about the next homotopy group in line: $\pi_5(\mathbb{S}^3)$. In this note, we present a proof that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ for some (constructively defined) $n \in \{1,2\}$. We have not yet been able to carry out a (terminating) normalisation of this number in a constructive proof assistant such as `Cubical Agda`, but we are hopeful – Pontrjagin and Whitehead only showed that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$ in 1950[1] [Pon50; Whi50], so we can let `Agda` think for another couple of years before we hit the 78-year mark.

Concretely, our first key contribution – Theorem 1 – is a HoTT version of a theorem by Gray [Gra73] which relates cofibres of certain so-called *Whitehead products* to the fibre of the *pinch map*. Using that such cofibres are well known to be related to the homotopy groups of spheres, we are able to use Theorem 1 to derive our second key contribution – Theorem 2 – which is a characterisation of $\pi_5(\mathbb{S}^3)$. While we appreciate that Theorem 1 is a somewhat niche result, we hope that Theorem 2 will be of interest to a broader audience of type theorists as it presents a so-called 'Brunerie number'. That is, it presents a constructively defined $n \in \{1,2\}$ whose normalisation in a constructive proof assistant like `Cubical Agda` would yield a proof of a non-trivial theorem: in this case, of the fact that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. We will discuss our struggles with actually normalising/computing this number. While the formalisation of the proof that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ is in its early stages, the construction of our new Brunerie number is independent and can be defined in `Cubical Agda` with little effort using the formalisation of $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ due to Ljungström and Mörtberg [LM23]. We remark that we do also have a pen-and-paper proof of the fact that our Brunerie number is 2. Nevertheless, we are still interested in producing a fully computer-assisted proof.

Preliminaries/notation We will not go into any detailed proofs and will not require too much knowledge of classical homotopy theory. We do, however, rely on some basics of HoTT which we briefly rush through here. The reader comfortable with HoTT can skim this section.

Pointed types and maps: a pointed type is a pair $(A, \star_A)$ where $\star_A : A$. We will simply write A and leave $\star_A$ implicit. We use the same convention for pointed maps and write $f : A \to_\star B$ for a pointed map from A to B and leave the proof $\star_f : f(\star_A) = \star_B$ implicit.

Loop spaces: given a pointed type A, we let $\Omega(A) := (\star_A = \star_A)$ denote its loop space. Given a pointed map $f : A \to_\star B$, we write $\Omega\, f : \Omega(A) \to_\star \Omega(B)$ for the functorial action of Ω.

Pushouts: given a span $B \xleftarrow{f} A \xrightarrow{g} D$, we can form its pushout. This is the higher inductive type (HIT) $P_{f,g}$ with constructors $\mathsf{inl} : B \to P_{f,g} \qquad \mathsf{inr} : D \to P_{f,g} \qquad \mathsf{push} : (a : A) \to f(a) = g(a).$ Pushouts are always taken to be pointed by $\mathsf{inl}(\star_B)$ when B is pointed. Important instances are:

Name	Notation	Pushout of span	Comments
Join	$A * B$	$A \leftarrow A \times B \to B$	
Cofibre (of f)	C_f	$\mathbb{1} \leftarrow A \xrightarrow{f} B$	
Wedge sum	$A \vee B$	$A \xleftarrow{x \mapsto \star_A} \mathbb{1} \xrightarrow{x \mapsto \star_B} B$	A and B are pointed types
Suspension	ΣA	$\mathbb{1} \leftarrow A \to \mathbb{1}$	When A is pointed, there is a 'suspension function' $\sigma_A : A \to \Omega(\Sigma A)$ given by $\sigma_A(x) := \mathsf{push}(x) \cdot \mathsf{push}(\star_A)^{-1}$

Recall also that the n-sphere can be defined in terms of suspension by defining it to be the $(n+1)$-fold suspension of the empty type, i.e. $\mathbb{S}^n := \Sigma^{n+1}\bot$.

[1]The original proofs concerned $\pi_6(\mathbb{S}^4)$ but this group is isomorphic to the one in question by the quaternionic Hopf fibration [BR18].

Homotopy groups: given a pointed type A, we define its nth homotopy group by $\pi_n(A) := \| \mathbb{S}^n \to_\star A \|_0$.[2] Here, $\| - \|_0$ denotes *set truncation*, i.e. the operation which takes a type and forces it to satisfy UIP. The type $\pi_n(A)$ has a natural group structure when $n \geq 1$ which is abelian when $n \geq 2$. Given a pointed map $f : A \to_\star B$, there is an induced homomorphism $f_* : \pi_n(A) \to \pi_n(B)$. This yields a long exact sequence $\cdots \to \pi_{n+1}(B) \to \pi_n(\mathsf{fib}_f) \xrightarrow{\mathsf{fst}_*} \pi_n(A) \xrightarrow{f_*} \pi_n(B) \to \pi_{n-1}(\mathsf{fib}_f) \to \dots$ where fib_f is short for $\mathsf{fib}_f(\star_B)$, the fibre of f over $\star_B$, i.e. $(a : A) \times (f(a) = \star_B)$.

Connectedness: a type A is n-connected if its n-truncation, $\| A \|_n$, is contractible. We say that a map $f : A \to B$ is n-connected if all its fibres are n-connected. The key thing we will need to know about connectedness is that if $f : A \to B$ is an n-connected and pointed map, then the induced map $f_* : \pi_m(A) \to \pi_m(B)$ is an isomorphism when $m \leq n$ and surjective when $m = n + 1$.

The pinch map: an important map for us will be the so-called pinch map. Given a function $f : A \to B$, we define $\mathsf{pinch}_f : C_f \to \Sigma A$ by

$$\mathsf{pinch}_f(\mathsf{inl}(\star_{\mathbb{1}})) := \mathsf{inl}(\star_{\mathbb{1}}) \quad \mathsf{pinch}_f(\mathsf{inl}(b)) := \mathsf{inr}(\star_{\mathbb{1}}) \quad \mathsf{ap}_{\mathsf{pinch}_f}(\mathsf{push}(a)) := \mathsf{push}(a).$$

The main results In order to state our main results, we will need a key construction from homotopy theory called *Whitehead products* [Ark62]. These turn homotopy groups into a graded Lie superalgebra and provide us with principled ways of constructing elements of homotopy groups. All you need to know for this presentation, however, is that (just like the original Brunerie number [Bru16]), the Brunerie number we present here will be defined in terms of a certain Whitehead product. The most bare-bones definition of the Whitehead product is called the *generalised Whitehead product*: given two functions $f : \Sigma A \to D$, $g : \Sigma B \to D$, we define their (generalised) Whitehead product to be the function $[f, g] : A * B \to D$ defined by

$$[f, g](\mathsf{inl}(a)) := \star_D \quad [f, g](\mathsf{inr}(b)) := \star_D \quad \mathsf{ap}_{[f,g]}(\mathsf{push}(a)) := (\Omega\, g)(\sigma_B(b)) \cdot (\Omega\, f)(\sigma_A(a))$$

This definition follows those of Brunerie and Ljungström & Mörtberg [Bru16; LM24].

It turns out that some homotopy groups of spheres correspond to homotopy groups of cofibres of certain Whitehead products. Indeed, Brunerie showed that $\pi_{k+1}(\mathbb{S}^{n+1}) \cong \pi_k(C_{[\mathsf{id}_{\mathbb{S}^n}, \mathsf{id}_{\mathbb{S}^n}]})$ for $k \leq 3n - 2$.[3] This allowed him to define his (in)famous Brunerie number and, later, characterise $\pi_4(\mathbb{S}^3)$. Here, we are interested in the next homotopy group, $\pi_5(\mathbb{S}^3)$ and, as it happens, this group too can be understood in terms of the cofibre of a Whitehead product. To get there, let us state our first main theorem – it is a HoTT counterpart of a classical theorem by Gray [Gra73] and relates the (homotopy groups of) cofibres of Whitehead products of the form $[\mathsf{id}_{\Sigma B}, f]$ to fibres of the pinch map.

Theorem 1. *Let A be an $(a-1)$-connected pointed type and B be any pointed type. Let $f : \Sigma A \to_\star \Sigma B$. In this case, there is a $2a$-connected map $\gamma : C_{[\mathsf{id}_{\Sigma B}, f]} \to \mathsf{fib}_{\mathsf{pinch}_f}$ where, recall, $\mathsf{pinch}_f : C_f \to \Sigma^2 A$.*

Assume further that B in Theorem 1 is $(b-1)$-connected. In this case, we get that $\pi_n(\mathsf{fib}_{\mathsf{pinch}_f}) \cong \pi_n(\Sigma B)$ when $n \leq a + b$ for elementary connectedness reasons. Combining this information with Theorem 1 and the long exact sequence associated with the fibration sequence $\mathsf{fib}_{\mathsf{pinch}_f} \to C_f \to \Sigma^2 A$, we obtain a new sequence of the form

$$\begin{gathered}\cdots \to \pi_{n+1}(\Sigma^2 A) \to \pi_n(F_n) \to \pi_n(C_f) \to \pi_n(\Sigma^2 A) \to \pi_{n-1}(F_{n-1}) \to \dots \qquad \text{where} \\ F_{n \leq a+b} := \Sigma B \qquad F_{a+b<n\leq 2a} := C_{[\mathsf{id}_{\Sigma B}, f]} \qquad F_{2a<n} := \mathsf{fib}_{\mathsf{pinch}_f}\end{gathered}$$

Let us instantiate this sequence with $A := \mathbb{S}^2$, $B := \mathbb{S}^1$, $f := [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]$, $a = 2$ and $b = 1$. Note that this is well-typed because the domain of f is $\mathbb{S}^1 * \mathbb{S}^1$ which is equivalent to $\mathbb{S}^3$. The conflation of these spaces will be used without comment from now on. We get

$$\pi_5(\mathbb{S}^4) \to \pi_4(C_{[\mathsf{id}_{\mathbb{S}^2}, [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]]}) \to \pi_4(C_{[\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]}) \to \pi_4(\mathbb{S}^4) \to \pi_3(\mathbb{S}^2) \to \pi_3(C_{[\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]}).$$

All of these groups have alternative descriptions.

- The first group, $\pi_5(\mathbb{S}^4)$, is isomorphic to $\pi_4(\mathbb{S}^3)$ by stability [UF13, Corollary 8.6.15], which we know is further isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

[2] We could equivalently have set $\pi_n(A) := \| \Omega^n(A) \|_0$. While this definition makes the group structure on $\pi_n(A)$ very clear (it is simply path composition), it makes some other constructions (most importantly for us, Whitehead products) somewhat more roundabout.

[3] Concretely, Brunerie shows in the proof of Proposition 3.4.4 that $J_2(\mathbb{S}^n) \simeq C_{[\mathsf{id}_{\mathbb{S}^n}, \mathsf{id}_{\mathbb{S}^n}]}$, where the former type denotes the second type in the *James construction* on $\mathbb{S}^n$ – a type which, by Brunerie's Proposition 3.2.1, has π_k isomorphic to $\pi_{k+1}(\mathbb{S}^{n+1})$ for $k \leq 3n - 2$.

- For the second group, we have that $[\mathsf{id}_{\mathbb{S}^2}, [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]] = [\mathsf{id}_{\mathbb{S}^2}, 2\eta] = 2[\mathsf{id}_{\mathbb{S}^2}, \eta] = 0$ where the first equality is (in essence) the original Brunerie number, the second follows from bilinearity of Whitehead products [Lju25] and the third follows from 2-torsion. Thus, $C_{[\mathsf{id}_{\mathbb{S}^2}, [\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]]}$ is the cofibre of the constant map $\mathbb{S}^4 \to_\star \mathbb{S}^2$ and is, as such, equivalent to $\mathbb{S}^5 \vee \mathbb{S}^2$. Furthermore, we have $\pi_4(\mathbb{S}^5 \vee \mathbb{S}^2) \cong \pi_4(\mathbb{S}^5) \times \pi_4(\mathbb{S}^2) \cong \pi_4(\mathbb{S}^2) \cong \pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$, where the second to last isomorphism comes from the Hopf map [Bru16, Proposition 2.6.8].
- The third and sixth groups, $\pi_n(C_{[\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]})$ for $n \in \{3, 4\}$, are isomorphic to $\pi_{n+1}(\mathbb{S}^3)$ by (as mentioned briefly earlier) Brunerie's work [Bru16, Section 3.4].
- The fourth and fifth groups, $\pi_4(\mathbb{S}^4)$ and $\pi_3(\mathbb{S}^2)$ respectively, are well known to be isomorphic to the integers [UF13, Theorem 8.6.17, Corollary 8.6.19].

With this information, we can rewrite this instance of the sequence as follows.

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{d} \mathbb{Z}/2\mathbb{Z} \to \pi_5(\mathbb{S}^3) \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$

It follows completely abstractly, i.e. without knowing any details about either of the maps in the above sequence, that the second map must be surjective, and thus we have constructed our Brunerie number:

Theorem 2. *$\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/(2 - d(1))\mathbb{Z}$.*

The situation we find ourselves in now should seem awfully familiar to anyone familiar with Brunerie's thesis... We need to compute $d(1) : \mathbb{Z}/2\mathbb{Z}$. By unfolding its definition, we see that this number can be understood as the result of applying the isomorphism $\phi : \pi_4(\mathbb{S}^2) \cong \mathbb{Z}/2\mathbb{Z}$ (in HoTT, due to Brunerie [Bru16] and formalised by Ljungström & Mörtberg [LM23]) to the composite map $\mathbb{S}^4 \xrightarrow{\Sigma\eta} \mathbb{S}^3 \xrightarrow{[\mathsf{id}_{\mathbb{S}^2}, \mathsf{id}_{\mathbb{S}^2}]} \mathbb{S}^2$ (viewed as an element of $\pi_4(\mathbb{S}^3)$) where $\eta : \pi_3(\mathbb{S}^2)$ is the generator. This composite map is equal to $\mathbb{S}^4 \xrightarrow{\Sigma\eta} \mathbb{S}^3 \xrightarrow{2\eta} \mathbb{S}^2$. Although the easiest approach would be to simply compute $d(1) := \phi(2\eta \circ \Sigma\eta)$ in `Cubical Agda`, we have not had much success.[4] It is, however, relatively easy to prove that $d(1) = 0$ by showing that $2\eta \circ \Sigma\eta = 0$ by hand. This is a consequence of the fact that the precomposition map $(-) \circ \Sigma\eta$ defines a group homomorphism $\pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^2)$ and therefore must vanish on 2η due to 2-torsion in $\pi_4(\mathbb{S}^2)$. Hence, we may conclude that $\pi_5(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$.

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