Extensional concepts in intensional type theory, revisited

Krzysztof Kapulkin and Yufeng Li

– 💥 – Background

Hofmann, Martin. Extensional constructs in intensional type theory. PhD thesis, 1995.

Kapulkin, Krzysztof and Lumsdaine, Peter LeFanu. The homotopy theory of type theories. Advances in Mathematics, 2018.

Isaev, Valery. Morita equivalences between algebraic dependent type theories. arXiv:1804.05045, 2020.

🖌 – Main result

Kapulkin, Krzysztof and Li, Yufeng. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

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Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p: Id_{\mathcal{A}}(a_1, a_2)$

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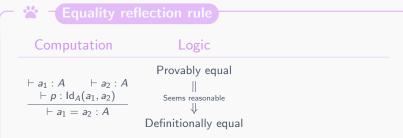
😤 – Equality reflection rule

Computation

$$\vdash a_1 : A \qquad \vdash a_2 : A \\ \vdash p : \mathsf{Id}_A(a_1, a_2) \\ \vdash a_1 = a_2 : A$$

Adding equality reflection gives extensional type theory (ETT).

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ComputationLogicTopologyLogicTopologyProvably equalProvably equa

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- Changing terms between types indexed by definitionally equal terms is proof-independent.
- Changing terms between types indexed by propositionally equal terms depends on the proof of equality.

 $\frac{\vdash p, p': \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p'): \mathsf{Id}(p, p')}$

Uniqueness of identity proofs

Homotopically discrete space





– 🛃 – Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p, p': \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p'): \mathsf{Id}(p, p')} \longleftrightarrow \frac{\vdash p: \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2: A}$$

Limitation. Syntactic result did not account for extensions.

🝽 – Definition

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😤 – Need to Determine

- 1. What is a model of a type theory?
- 2. A suitable notion of equivalence between categories of models?

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7/14

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Grading Truncation

$$\operatorname{ob} \mathbb{C} = \coprod_{n \in \mathbb{N}} \operatorname{ob}_n \mathbb{C} \quad \operatorname{ob}_{n+1} \mathbb{C} \xrightarrow{\operatorname{ft}} \operatorname{ob}_n \mathbb{C}$$

Notation. If ft $A = \Gamma$ we write $A = \Gamma.A$.

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Projection

$$\Gamma.A \xrightarrow{\pi} \Gamma$$

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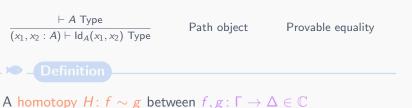
Grading Truncation $ob \mathbb{C} = \prod_{n \in \mathbb{N}} ob_n \mathbb{C}$ $ob_{n+1} \mathbb{C} \xrightarrow{ft} ob_n \mathbb{C}$ Projection

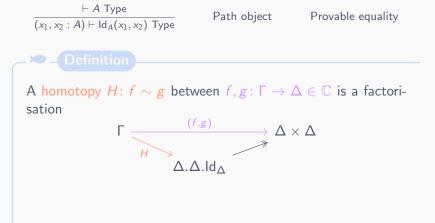
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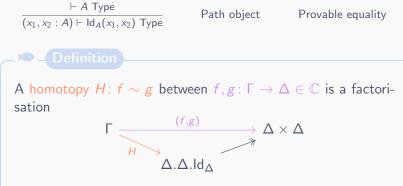
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Substitutions

$$\begin{array}{c} \Delta.f^*A \xrightarrow{f.A} \Gamma.A \\ \pi \downarrow \qquad \qquad \downarrow \pi \\ \Delta \xrightarrow{f} \Gamma \end{array}$$







Homotopy equivalences $w \colon \Gamma \to \Delta$ are those maps admitting left and right homotopy inverses.



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$$\begin{array}{cccc}
\mathbb{D} & \overline{A} \\
\downarrow_{F} & \downarrow \\
\mathbb{C} & F\overline{A} & - \xrightarrow{\simeq} & A
\end{array}$$

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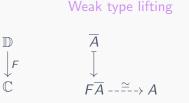
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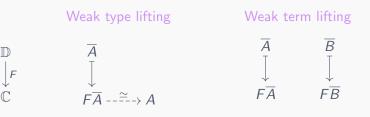
 \overline{R}



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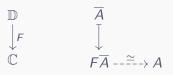
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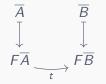
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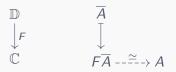


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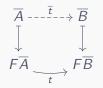
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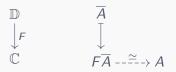


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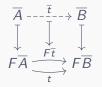
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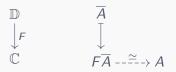


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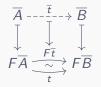
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Example (Isaev 2020). The type theories ITT+Unit and ITT+Contr are Morita equivalent.



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Quillen equivalence by definition says that the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cofibrant models is a weak equivalence.

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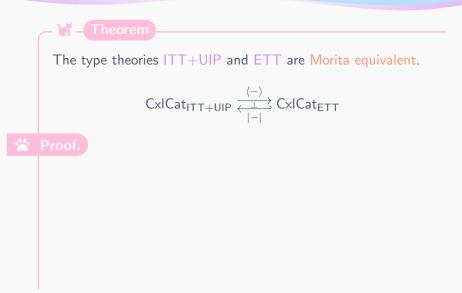
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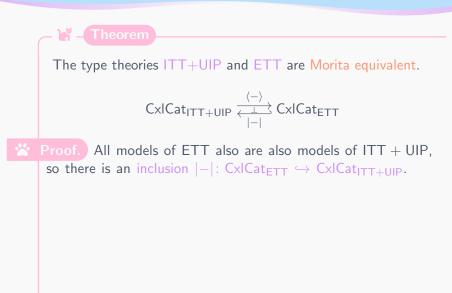
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- …then the expressible and provable statements in those two models are correspond propositionally within type theory.





<mark>רא</mark> – Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

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$$\mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}} \xleftarrow{\langle -\rangle}{\leftarrow \perp} \mathsf{CxlCat}_{\mathsf{ETT}}$$

Proof. All models of ETT also are also models of ITT + UIP, so there is an inclusion |-|: CxlCat_{ETT} \hookrightarrow CxlCat_{ITT+UIP}. By cocompleteness, it has a left adjoint $\langle - \rangle$.

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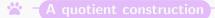
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From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}}$ to $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$



 To support equality reflection: must identify homotopic maps. 13/14

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😤 – A quotient construction

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 - ► Example. The map Bool → Bool swapping true and false is a propositional isomorphism but is not the identity even under equality reflection.
- ► Upshot. (C) is obtained from C by carefully choosing a wide subcategory of homotopy equivalences to collapse.



🗕 😤 – Future directions

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- Encompassing internal universes.
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Thank you!