Löb's Theorem and Provability Predicates in Rocq

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COMPUTER SCIENCE

Inria

Introduction

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  - $S \vdash \varphi$ iff $S \vdash \mathrm{Pr}(\overline{\varphi})$

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- Many proof techniques known to fail
- Can a synthetic perspective simplify arguments?
  - $\rightarrow$ Usually, technically intricate details vanish, up to 90% shorter proofs

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‘Sufficiently strong’ provability predicates:

Hilbert-Bernays-Löb (HBL) Conditions (Hilbert-Bernays (1939), Löb (1955))

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HBL + Diagonalisation property = Löb's theorem (abstract argument)

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Axiom ($\mathsf{CT}_{\mathsf{PA}}$, Hermes and Kirst (2022))

For all $f : \mathbb{N} \to \mathbb{N}$ there is $\varphi_f(x_1, x_2) : \mathbb{F}$ such that for all $n : \mathbb{N}$,

$$\mathsf{PA} \vdash \forall y.\, \varphi_f(\overline{n}, y) \leftrightarrow y = \overline{f\,n}.$$

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[3]See also Pédrot (2024), Swan and Uemura (2019)

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Corollary

There is $\mathrm{Pr}_{\mathsf{CT}}(x) : \mathbb{F}$ *such that* $\mathsf{PA} \vdash \varphi$ *iff* $\mathsf{PA} \vdash \mathrm{Pr}_{\mathsf{CT}}(\overline{\varphi})$*.*

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Problem

$\mathrm{CT}_{\mathrm{PA}}$ not strong enough for Löb's theorem (internal vs external provability).

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- Proof '=' List of formulas
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2. mechanised necessitation as well as the distributivity law for it.

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Is there a proof of Löb's theorem à la Kirst and Peters? No!

- Mechanised proof of Löb's theorem
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  - ➤ In Isabelle based on Paulson's development, axiom-free

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- Mechanised diagonal lemma and key limitative theorems assuming CT_{PA}

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- Analysed why CT_{PA} is too weak for Löb's theorem
- Mechanised extension of PA easing definition of internal provability predicates
- Gave candidate for internal provability predicate and parts of correctness proof

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Mechanisation

Rocq

- 2600 lines of code (600 specification, 1900 proof, 100 comment)
- Most intricate proof: Distributivity law in EHA (about 400 lines of code)
- Koch's [HKK21] proof mode immensely helpful
- Lots of code dealing with substitutions

Isabelle

- 100 lines of code (60 for Löb proof, 40 for lemmas)
- Can still be shortened

Background: Used Hilbert System

$\mathcal{H}(\varphi \to \psi \to \varphi)$	$\mathcal{H}((\varphi \to \psi \to \tau) \to (\psi \to \tau) \to \varphi \to \tau)$	
$\mathcal{H}(\varphi \to \psi \to \varphi \wedge \psi)$	$\mathcal{H}(\varphi \wedge \psi \to \varphi)$	
$\mathcal{H}(\varphi \to \varphi \vee \psi)$	$\mathcal{H}(\varphi \wedge \psi \to \psi)$	
$\mathcal{H}(\psi \to \varphi \vee \psi)$	$\mathcal{H}(\varphi \vee \psi \to (\varphi \to \tau) \to (\psi \to \tau) \to \tau)$	
$\mathcal{H}(\bot \to \varphi)$	$\mathcal{H}(\varphi \to \forall x.\, \varphi)$	x fresh for φ
$\mathcal{H}((\forall x.\, \varphi) \to \varphi[x \mapsto t])$	$\mathcal{H}((\forall x.\, \varphi \to \psi) \to (\forall x.\, \varphi) \to \forall x.\, \psi)$	
$\mathcal{H}(\varphi[x \mapsto t] \to \exists x.\, \varphi)$	$\mathcal{H}((\exists x.\, \varphi) \to (\forall x.\, \varphi \to \psi) \to \psi)$	x fresh for ψ

$$\frac{\mathrm{PA} \vdash_{\mathcal{H}} \varphi \to \psi \qquad \mathrm{PA} \vdash_{\mathcal{H}} \varphi}{\mathrm{PA} \vdash_{\mathcal{H}} \psi} \qquad \frac{\varphi \in \mathcal{H}}{\mathrm{PA} \vdash_{\mathcal{H}} \forall x_1. \ldots x_n.\, \varphi} \qquad \frac{\varphi \in \mathrm{PA}}{\mathrm{PA} \vdash_{\mathcal{H}} \varphi}$$

Elements from Rautenberg, Troelstra and Schwichtenberg, as well as both.

Extended PA

Definition (Extended Signature of Peano Arithmetic (EPA), simplified)

In addition to the symbols of PA, EPA contains the following function symbols:

$[\,]$ (nil)	$\lvert\ell\rvert$ (length)	$\ell +\!\!+\, \ell'$ (append)
$x :: \ell$ (cons)	$\ell[i]$ (indexed access)	$x \rightsquigarrow y$ (implication)

Further, EPA adds the unary predicate symbol $\mathcal{A}$ to PA.

- $\mathrm{EPA} \vdash \overline{\varphi \to \psi} = \overline{\varphi} \rightsquigarrow \overline{\psi}$ (object level implication function)
- If $\varphi \in \mathcal{H}$, then $\mathrm{EPA} \vdash \mathcal{A}\,(\forall x_1. \ldots x_n. \varphi)$
- If $\varphi \in \mathrm{PA}$, then $\mathrm{EPA} \vdash \mathcal{A}\,\varphi$

Formal proofs: Spelling out (some of) the Details

Definition (Formal proofs)

A proof of φ is a nonempty list $\ell = [\psi_1, \ldots, \psi_n] : \mathcal{L}(\mathbb{F})$ with $\varphi = \psi_n$ s.t. for each i

- ψ_i is an axiom of PA, a generalisation of a Hilbert axiom, or
- there are $j, j' < i$ such that ψ_i follows from $\psi_j, \psi_{j'}$ by modus ponens.

Definition (Provability predicate)

$$\begin{aligned}
\mathrm{Prf}(x, y) &:= (\exists z.\, |x| = S\, z \wedge x[z] = y) \wedge \forall i.\, i < |x| \to \mathrm{WellFormed}(x, i) \\
\mathrm{WellFormed}(x, i) &:= \mathcal{A}(x) \vee \exists j\, j'.\, j < i \wedge j' < i \wedge x[j] = x[j'] \rightsquigarrow x[i]
\end{aligned}$$

Technical Background: Gödel Numberings

Problem

Let $\varphi(x), \psi : \mathbb{F}$.
We used $\varphi(\overline{\psi})$ for 'substituting some encoding of ψ for x in φ'.
ψ is not a **number**, but a **formula**.

Typical issue. Gödel faced it himself.

Remark (Gödelisation)

There are functions $\text{göd} : \mathbb{F} \to \mathbb{N}$, $\text{göd}^{-1} : \mathbb{N} \to \mathbb{F}$ *inverting each other.*

$\varphi(\overline{\psi}) \rightsquigarrow \varphi(\overline{\text{göd}(\psi)})$

Technical Background: $\mathsf{CT}_{\mathsf{PA}}$ is too Weak

Axiom ($\mathsf{CT}_{\mathsf{PA}}$)

For every $f : \mathbb{N} \to \mathbb{N}$, there is a formula $\varphi(x_1, x_2)$ such that for all $n : \mathbb{N}$

$$\mathsf{PA} \vdash \forall y.\, \varphi(\overline{n}, y) \leftrightarrow y = \overline{f\ n}.$$

Example

Suppose the successor function $\mathtt{S} : \mathbb{N} \to \mathbb{N}$ is represented by $\varphi_{\mathtt{S}}(x, y)$.

Question: Can we derive, for all $n \in \mathbb{N}$, that $\mathsf{PA} \vdash \varphi_{\mathtt{S}}(\overline{n}, \mathsf{S}\,\overline{n})$? **Yes!**

- Use property of $\varphi_{\mathtt{S}}$: $\mathsf{PA} \vdash \mathsf{S}\,\overline{n} = \overline{\mathtt{S}n}$
- By definition of numerals, $\mathsf{PA} \vdash \mathsf{S}\,\overline{n} = \mathsf{S}\,\overline{n}$, easy to finish

Question: Can we derive $\mathsf{PA} \vdash \forall x.\, \varphi_{\mathtt{S}}(x, \mathsf{S}\,x)$? **No!**

- Introduce x: $\mathsf{PA} \vdash \varphi_{\mathtt{S}}(x, \mathsf{S}\,x)$. No way to continue as x not a numeral